Atiyah's Proof of the Convexity Theorem

John R. Calabrese

Abstract

Following A. Cannas da Silva's book [CdS01] we give M. Atiyah's proof of the Atiyah-Guillemin-Sternberg theorem. In the first section we recall the essential definitions and facts from differential geometry and in the second section we state and prove the theorem.

1 Preparing for the Theorem

First let us introduce three of the main protagonists: riemannian metrics, symplectic forms and almost complex structures.

1.1 Definitions

Let M be a smooth manifold.

- *A **Riemannian Metric** is a smooth field g of positive definite scalar products:*
$$g_p : T_pM \times T_pM \to \mathbb{R} \quad \forall p \in M.$$
- *A **Symplectic Form** is a smooth field ω of skew-symmetric bilinear maps:*
$$\omega_p : T_pM \times T_pM \to \mathbb{R} \quad \forall p \in M,$$
such that ω is closed as a 2-form (i.e. $d\omega = 0$).
- *An **Almost Complex Structure** is a smooth field J of linear endomorphisms:*
$$J_p : T_pM \to T_pM \quad \forall p \in M[1],$$
such that $J_p^2 = -\mathrm{id}_{T_pM}$, for all $p \in M$.

The couples (M, g), (M, ω) and (M, J) will be called riemannian, symplectic and almost complex manifolds respectively.

*If (M, ω) is a symplectic manifold then an almost complex structure J on M is said to be **compatible** with ω if (M, g), where g is defined by*
$$T_pM \times T_pM \ni (v, w) \mapsto g_p(v, w) = \omega(v, J_p w) \in \mathbb{R} \quad \forall p \in M,$$
*is a riemannian manifold. In this case we speak of **compatible triples** (M, ω, J, g).*

[1]It might be more satisfactory to view these three objects as sections of appropriate vector bundles over M, but this point of view is not of our concern here.

Given a symplectic manifold it is always possible to find compatible almost complex structures on it. In general these structures are not unique, but if we fix a riemannian metric we have a 'canonical'[2] choice (cfr. [CdS01] p. 68).

1.1 Exps

Let us start with Lie groups. Let G be a Lie group. By a **one-parameter subgroup** of G we mean a Lie group homomorphism from $\mathbb{R}$ to G. We recall that one-parameter subgroups of G and the tangent space T_eG of G to the identity are in one-to-one correspondence. This correspondence is given by (cfr. [War83] chap. 3):

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathbb{R}, G) \ni \vartheta \mapsto \left.\frac{d}{dt}\right|_0 \vartheta(t) \in T_eG.$$

1.2 Definition
*Let G be a Lie group. We define the **Lie group exponential map** to be:*

$$T_eG \ni X \mapsto \exp X = \vartheta(1) \in G$$

where ϑ is the one-parameter subgroup of G corresponding to X.

We shall use the following results.

1.3 Proposition
The exponential map is smooth and natural, i.e.:

$$\begin{tikzcd} T_eG \arrow[r, "T_e\phi"] \arrow[d, "\exp"'] & T_{e'}G' \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & G' \end{tikzcd}$$

where $\phi \in \mathrm{Hom}_{Lie}(G, G')$ is a Lie group homomorphism. Moreover if $X \in T_eG$ is a tangent vector at the identity of G then $\exp(t+s)X = (\exp tX)\cdot(\exp sX)$ for all $t, s \in \mathbb{R}$, where $\cdot$ indicates the Lie group product of G.

We now turn to riemannian manifolds. Here we have an analogous exponential map (cfr. [Lee97] p. 72–78, 89).

1.4 Definition
*Let (M, g) be a riemannian manifold and let $p \in M$ be a point of M. The **riemannian exponential map starting at** p is defined by:*

$$T_pM \ni X \mapsto \exp_p X = \gamma_X(1) \in M$$

where γ_X is the unique geodesic starting at p with tangent vector X.

[2] Throughout this article we will use the terms *natural* and *canonical* informally, since we won't have any opportunity of dealing with categorial issues.

We recall the following facts.

1.5 Proposition
The exponential map is smooth and natural:

$$\begin{tikzcd} T_pM \arrow[r, "T_p\phi"] \arrow[d, "\exp_p"'] & T_{p'}M' \arrow[d, "\exp_p"] \\ M \arrow[r, "\phi"] & M' \end{tikzcd}$$

where ϕ is an isometry (i.e. $\phi^* g' = g$). Moreover if $X \in T_pM$ is a tangent vector in p then $\exp_p tX = \gamma_X(t)$ for $t \in \mathbb{R}$.[3]

Another important result is the existence of normal coordinates.

1.6 Proposition
There exist U and V open neighborhoods respectively of $0 \in T_pM$ and $p \in M$ such that $\exp_p : U \to V$ is a diffeomorphism. Furthermore if we fix an orthonormal basis of T_pM we obtain an isomorphism F with $\mathbb{R}^n$ which in turn can be combined with $\exp_p^{-1}$ to give a coordinate chart $(V, F \circ \exp_p^{-1})$ which is called a ***normal coordinate chart****. Finally, if $(V, (x_i))$ is a normal coordinate chart on (M, g) and if $X = \sum_i X_i \partial_i \in T_pM$ then the local coordinate expression of $\exp_p$ is given by*

$$\exp_p tX = (tX_1, \dots, tX_n).$$

1.7 Proposition
Let G be a Lie group equipped with a bi-invariant riemannian metric. Then the Lie group exponential map is precisely the riemannian exponential map starting at the identity.

1.2 Morse-Bott Theory

What follows is just a glimpse of Morse-Bott theory, which plays an essential role in Atiyah's proof of the convexity theorem ([MS95], p. 177–179).

Let $f \in \mathscr{C}^\infty(M)$ be a smooth real-valued function on a riemannian manifold M. We write $S_p(f) : T_pM \to T_pM$ to indicate the linear map obtained from the hessian of f via the riemannian metric, for all $p \in M$.

1.8 Definitions
Let M be a compact connected riemannian manifold. A smooth function $f \in \mathscr{C}^\infty(M)$ on M is a ***Morse-Bott function*** *if $\operatorname{Crit} f$ decomposes into a finite disjoint union of connected submanifolds*[4] *of M (called* ***critical submanifolds****) and $T_p \operatorname{Crit} f = \ker S_p(f)$ for all $p \in \operatorname{Crit} f$.*

[3] To be honest we've just cheated in this last statement because $\exp tX$ isn't always defined for all $t \in \mathbb{R}$.

[4] By *submanifold* we mean a closed embedded submanifold

If f is Morse-Bott then

$$T_pM = T_p \operatorname{Crit} f \oplus E_p^- \oplus E_p^+,$$

where $E_p^\mp$ are the negative and positive (relatively to the hessian) subspaces of T_pM.

*The **index** of a critical submanifold C is $n^-(C) = \dim_R E_p^-$, where $p \in C$ is any point of C. Similarly the **coindex** of a critical submaifold is $n^+(C) = \dim_R E_p^+$.*

We have the following the result.

1.9 Lemma

Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact riemannian manifold whose critical submanifolds have all index and coindex different from one. Then the level sets of f are connected.

2 The Convexity Theorem

Before we can even state the theorem we need some definitions.

2.1 Definitions

*Let $\phi : (M, \omega) \to (M', \omega')$ be a smooth function between two symplectic manifolds. We say that ϕ is a **symplectomorphism** if it is a diffeomorphism and if $\phi^* \omega' = \omega$.*

*A **symplectic action** ψ of a Lie group G over a symplectic manifold (M, ω) is a smooth action $\psi : G \to \operatorname{Sympl}(M, \omega) < \operatorname{Diff}(M)$.*

We now want to define what a hamiltonian action is. Before we can do that we must clear some notation out.

If G is a Lie group we write $\mathfrak{g}$ to indicate the Lie algebra of left-invariant vector fields, which is canonically isomorphic to T_eG. If $X \in \mathfrak{g}$ is a left-invariant vector field (or equivalently a tangent vector at e) we may consider the one-parameter subgroup of G

$$\left\{\exp tX \;\middle|\; t \in \mathbb{R}\right\}.$$

If $\psi : G \to \operatorname{Sympl}(M, \omega)$ is a symplectic action we may compose ψ with $\exp tX$ to obtain $\psi_{\exp tX}$, a smooth one-parameter subgroup of symplectomorphisms. By differentiation we finally get

$$X_p^\# = \left.\frac{d}{dt}\right|_0 \psi_{\exp tX}(p) \in T_pM$$

which is a vector field on M, associated to X.

2.2 Definition

*A symplectic action $\psi : G \to \operatorname{Sympl}(M, \omega)$ is called **hamiltonian** if there exist a smooth map $\mu : M \to \mathfrak{g}^*$ (called the **moment map**) such that:*

- *If $X \in \mathfrak{g}$ is a left-invariant vector field on G and if we call μ^X the component of μ along X, i.e. the map given by:*

$$\mu^X(p) = \langle \mu(p), X \rangle = \mu(p)(X),$$

we have

$$d\mu^X = i_{X^\#}\omega = \omega(X^\#, \cdot).$$

- *The map μ is equivariant with respect to ψ and to the coadjoint action of G, i.e.:*

$$\mu \circ \psi_g = \mathrm{Ad}^*_g \circ \mu.$$

A remark is in order.

2.3 Remark *If G is abelian then the coadjoint action is trivial, so the second request is simply for μ to be invariant under the action.*

2.4 Remark *If we choose a basis of the tangent space $T_e G$ we have an isomorphism between $\mathfrak{g}$ and $\mathbb{R}^{\dim G} \simeq (\mathbb{R}^{\dim G})^*$. Hence we might regard $\mu : M \to R^{\dim G}$ as a map into a euclidean space.*

2.1 The Convexity Theorem

In what follows we will be concerned with hamiltonian actions of the torus. We choose coordinates $(\theta_1, \dots, \theta_m)$ on the m-torus (these coordinates are simply local sections of the standard covering map $\mathbb{R}^m \overset{e^{2\pi i \cdot}}{\to} T^m$), and we will deliberately confuse points of the torus with their coordinates. We also identify $T_0 T^m$ with $\mathbb{R}^m$ by means of the basis $\{\partial_{\theta_1}, \dots, \partial_{\theta_m}\}$

We can now finally state the theorem:

2.5 Theorem (Atiyah-Guillemin-Sternberg)
Let (M, ω) be a compact connected symplectic manifold and let $\psi : T^m \to \mathrm{Sympl}(M, \omega)$ be a hamiltonian action of the m-torus T^m over (M, ω) with moment map $\mu : M \to \mathbb{R}^m$. Then the level sets of the moment map are connected, the image of the moment map is convex, in fact it is the convex hull of the image of the fixed points of the action. In other words:

(1) $\mu^{-1}(\xi)$ *is connected, for all* $\xi \in \mathbb{R}^m$,

(2) $\mu(M)$ *is convex,*

(3) $\mu(M) = \mathrm{Conv}\, \mu(\mathrm{Fix}\, \psi)$,

where, if $A \subset \mathbb{R}^m$, $\mathrm{Conv}\, A$ indicates the convex hull of A.

Proof
The proof will be divided into several steps.

Step 1 There exists an almost complex structure J on M, compatible with ω, invariant under ψ, i.e.:

$$T_p\psi_\theta \circ J_p = J_{\psi_\theta} \circ T_p\psi_\theta \quad \forall p \in M, \theta \in T^m.$$

To find such a J we start with any given riemannian metric g and average it over the torus through ψ. This average is defined by:

$$\overline{g} = \int_{T^m} \psi_\theta^* g d\theta.$$

It is easy to see that ψ_θ is an isometry, with respect to $\overline{g}$, for all $\theta \in T^m$. Repeating the construction done in Cannas da Silva's book [CdS01] starting with $\overline{g}$ we obtain an almost complex structure J, compatible with ω. It is straightforward to check that J commutes with ψ.

From this point forth we endow M with the compatible (with respect to ω and J) riemannian metric $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. We point out that ψ_θ is an isometry for all $\theta \in T^m$.

Step 2 For any subgroup $G < T^m$, the fixed-point set for G

$$\operatorname{Fix} G = \bigcap_{\theta \in G} \operatorname{Fix} \psi_\theta$$

is a finite disjoint union of connected compact symplectic submanifolds of (M, ω).

Let $p \in \operatorname{Fix} G$. By the naturality of the geodesic exponential map we obtain the following commutative diagram:

$$\begin{tikzcd} T_pM \arrow[r, "T_p\psi_\theta"] \arrow[d, "\exp_p"'] & T_pM \arrow[d, "\exp_p"] \\ M \arrow[r, "\psi_\theta"] & M \end{tikzcd}$$

for every $\theta \in G$. Choosing normal neighbourhoods $\mathcal{U}, \mathcal{V}$ of $0 \in T_pM$ and $p \in M$ respectively yields this other diagram:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "T_p\phi"] \arrow[d, "\exp_p"'] & \mathcal{U}' \arrow[d, "\exp_p"] \\ \mathcal{V} \arrow[r, "\phi"] & \mathcal{V}' \end{tikzcd}$$

where $\exp : \mathcal{U} \to \mathcal{V}$ and $\exp : \mathcal{U}' \to \mathcal{V}'$ are diffeomorphisms.

It is now a simple matter to verify that we have a correspondence

$$\mathcal{U} \cap W_p \overset{\exp}{\longleftrightarrow} \mathcal{V} \cap \operatorname{Fix} G,$$

where

$$W_p = \bigcap_{\theta \in G} \ker \left(\mathrm{id}_{T_p M} - T_p \psi_\theta \right),$$

given by the exponential map. We now want to equip $\operatorname{Fix} G$ with a smooth atlas. Let

$$\{\epsilon_1, \ldots, \epsilon_{n_p}, \ldots, \epsilon_n\}$$

be an orthonormal basis of $T_p M$, where $n_p = \dim W_p$ and the first n_p vectors span W_p. This basis yields an isomorphism $E : T_p M \to \mathbb{R}^n$ which in turn gives the diffeomorphism

$$\phi = E \circ \exp^{-1} : \mathcal{V} \to E(\mathcal{U}).$$

Finally, it is easy to see that

$$\mathrm{pr}_{n_p+1} \circ \phi = \ldots = \mathrm{pr}_n \circ \phi = 0$$

is a slice for $\operatorname{Fix} G$. Since the function

$$\operatorname{Fix} G \ni p \longmapsto n_p \in \mathbb{Z}$$

is a locally constant function, we obtain that the connected components of $\operatorname{Fix} G$ are embedded connected submanifolds of M.

Considering the slice charts defined above we easily see that

$$T_p \operatorname{Fix} G = W_p.$$

Noticing that W_p is invariant under the almost complex structure J we conclude that W_p is a symplectic subspace of $T_p M$. Therefore $\operatorname{Fix} G$ is symplectic.

We now notice that $\operatorname{Fix} \psi_\theta$ is a closed subset of M[5], therefore $\operatorname{Fix} G$ is an intersection of closed sets and therefore himself a closed set. It's worth noticing that $\operatorname{Fix} G$ is also compact, being a closed subset of a compact space. Indeed we can say that the components of $\operatorname{Fix} G$ are closed embedded submanifolds of M. Since $\operatorname{Fix} G$ is a compact locally connected space it has finitely many components[6] so we finally have that

$$\operatorname{Fix} G = C_1 \cup \ldots \cup C_N$$

where the C_js (the components of $\operatorname{Fix} G$) are compact connected embedded symplectic submanifolds of M.

[5] This is a general fact: if f is a homeomorphism of a Hausdorff space X, then $\operatorname{Fix} f$ is closed in X.

[6] In general, if X is a locally connected space, the components of X are both open and closed. Furthermore, if X is also compact, we may choose the open covering given by the connected components. The compactness of X proves the claim that X has only finitely many components.

Step 3 If $X \in \mathbb{R}^m$ we have

$$\operatorname{Crit} \mu^X = \bigcap_{\theta \in T^X} \operatorname{Fix} \psi_\theta = \operatorname{Fix} T^X$$

where T^X is the closure of the one-parameter subgroup generated by X in T^m.

We start by considering the following chain of equivalent statements:

$$\begin{aligned} p \in \operatorname{Crit} \mu^X &\iff (d\mu^X)_p = 0 \\ &\iff \omega_p\left(X_p^\#, \cdot\right) = 0 \\ &\iff X_p^\# = 0 \\ &\iff \frac{d}{dt}\bigg|_{t=0} \psi_{\exp tX}(p) = 0. \end{aligned}$$

Now, if $p \in \operatorname{Crit} \mu^X$ and if $s \in \mathbb{R}$ is a point of the real line we have:

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=s} \psi_{\exp tX}(p) &= \frac{d}{dt}\bigg|_{r=0} \psi_{\exp(r+s)X}(p) \\ &= \frac{d}{dt}\bigg|_{r=0} \psi_{\exp sX} \circ \psi_{\exp tX}(p) \\ &= T_p\psi_{\exp sX} \circ T_{r=0}\psi_{\exp rX}(p)[1] \\ &= T_p\psi_{\exp sX}\left[\frac{d}{dt}\psi_{\exp rX}(p)\bigg|_{r=0}\right] \\ &= T_p\psi_{\exp sX}\left[X_p^\#\right] \\ &= 0. \end{aligned}$$

So the function:

$$\mathbb{R} \ni t \longmapsto \psi_{\exp tX}(p) \in M$$

is the constant map with image p. Therefore, if θ is an element of the one-parameter subgroup generated by X, $\psi_\theta(p) = p$. By continuity, p is fixed by all elements of T^X (i.e. $p \in \operatorname{Fix} T^X$).

Now, if $p \operatorname{Fix} T^X$, it is straight forward to check that $X_p^\# = 0$ and therefore that p is a critical point for μ^X (i.e. $p \in \operatorname{Crit} \mu^X$).

Step 4 If $X \in \mathbb{R}^m$, μ^X is a Morse-Bott function with even-dimensional critical submanifolds of even index and coindex.

We already now that $\operatorname{Crit} \mu^X$ has a nice decomposition into submanifolds. We now want to shaw that:

$$T_p \operatorname{Crit} \mu^X = \bigcap_{\theta \in T^X} \ker\left(\mathrm{id}_{T_pM} - T_p\psi_\theta\right) = \ker S_p$$

where S_p is the linear map obtained from the hessian of μ^X via the riemannian metric. We have already shown the first equality. Let's consider now this identity:

$$(\star) \qquad T_p\psi_{\exp tX} = e^{-tJ_pS_p}$$

which we will hopefully prove someday. Through identity ($\star$) we see that vectors in the kernel of S_p correspond to eigenvectors of eigenvalue 1 of $T_p\psi_{\exp tX}$.

Since the positive and negative spaces of the hessian are invariant under the almost complex structure we conclude that the critical submanifolds of ψ have all even (therefore different from one) index and coindex.[7]

Step 5 The action ψ is called **effective** if the 1-forms $d\mu_1, \ldots, d\mu_m$, obtained by the corresponding moment map $\mu = (\mu_1, \ldots, \mu_m)$, are linearly independent over $\mathbb{R}$. If the action is not effective then it reduces to that of an $(m-1)$-torus.

2.6 Remark *If ψ is effective then μ^X is non-constant, for all $X \in \mathbb{R}^m$.*

We now proceed by induction over the dimension of the torus. Let us consider the following statements:

A_m: The level sets of a moment map of a hamiltonian action of the m-torus T^m are connected.

B_m: The image of a moment map of a hamiltonian action of the m-torus T^m is convex.

A We begin by proving A.

The statement A_1 is precisely Lemma 1.9 since $\mu = \mu^1$ is a Morse-Bott function with indexes and coindexes different from one.

Let us assume now that A_k is true for $k < m$. We con now also assume that ψ is effective.

Step 6 Let $\mathcal{C}$ be:

$$\mathcal{C} = \operatorname{Crit}\mu = \bigcup_{X\neq 0} \operatorname{Crit}\mu^X.$$

We want to show that:

$$\mathcal{C} = \bigcup_{0\neq X\in\mathbb{Z}^m} \operatorname{Crit}\mu^X.$$

One inclusion is obvious. Let p be a critical point of μ. There must be $X \in \mathbb{R}^m$ such that $(d\mu^X)_p = 0$. There must also be $X' \in \mathbb{Z}^m$ such that $(d\mu^X)_p = 0$.[8]

Now we want to see that $\mathcal{C}$ is closed. Consider:

[7] I admit it, I haven't proven anything. But neither does Ana.

[8] This should be a consequence of the fact that a closed one-parameter subgroup of the torus can (I think) always be generated by a vector with integer components.

$$\begin{tikzcd} S^n \times M \arrow[r, "f"] \arrow[d, "\mathrm{pr}_2"'] & \mathbb{R} \\ M & \end{tikzcd}$$

where $f(X, p) = \mu^X(p)$, which is continuous. We notice that $\mathcal{C} = \mathrm{pr}_2 \circ f^{-1}(0)$, which is compact, hence closed in M.

Now, if $X \neq 0$ and $\theta \in T^X$, $M \setminus \mathrm{Fix}\, \psi_\theta$ is an open dense subset of M. Recalling that Fix T^X is a finite disjoint union of submanifolds we have that $M \setminus T^X$ is also a dense open subset of M. Since we have

$$\begin{aligned} M \setminus \mathcal{C} &= M \setminus \bigcup_{0 \neq X \in \mathbb{Z}^m} \mathrm{Crit}\, \mu^X \\ &= M \setminus \bigcup_{0 \neq X \in \mathbb{Z}^m} \mathrm{Fix}\, T^X \\ &= \bigcap_{0 \neq X \in \mathbb{Z}^m} M \setminus \mathrm{Fix}\, T^X, \end{aligned}$$

we may affirm, by Baire's category theorem, that $M \setminus \mathcal{C}$ is dense in M. Since we already know that $\mathcal{C}$ is closed, we have that $M \setminus \mathcal{C}$ is an open dense subset of M.

Now, using the inductive hypothesis, we should see that the regular values of μ are dense in $\mu(M)$ and that the preimage of a regular value is connected. Finally, using continuity and density, we should be able to see that the preimage of any $\xi \in \mathbb{R}^m$ is connected. Something like:

Step 7 The preimage of a regular value is connected.

$\mu = (\mu_1, \ldots, \mu_m)$, $\xi = (\xi_1, \ldots, \xi_m)$ regular value for μ.

This proves A_m for every $m \in \mathbb{N}^\times$.

$$Q = \bigcap_{j=1}^{m-1} \mu_j^{-1}(\xi_j)$$

e

$$\mu^{-1}(\xi) = Q \cap \mu_m^{-1}(\xi_m)$$

Q is connected by the induction hypothesis. If we show that

$$\mu_m : Q \to \mathbb{R}$$

is Morse-Bott with index and coindex different from one we win, since by the lemma we know that the level sets will be connected.

B We now turn to B.

The statement B_1 is trivial because in $\mathbb{R}$ connectedness is equivalent to convexity.

Let us assume now that B_k is true for $k < m$. Choose an injective matrix $A \in \mathbb{Z}^{m \times m-1}$, which can be regarded as an injective map between the tori T^{m-1} and T^m. We can consider the action:

$$T^{m-1} \ni \theta \stackrel{A}{\longmapsto} \psi_\theta = \psi_{A\theta} \in \mathrm{Sympl}(M, \omega)$$

which is a hamiltonian action with moment map

$$M \ni p \mapsto \mu_A(p) = A^T \mu(p) \in \mathbb{R}^{m-1}.$$

Let $\xi \in \mathbb{R}^{m-1}$ be a point in $\mathbb{R}^{m-1}$ and let $p_0 \in \mu_A^{-1}(\xi)$ a point in the level set of ξ. Then

$$p \in \mu_A^{-1}(\xi) \iff A^T \mu(p) = \xi = A^T \mu(p_0) \iff \mu(p) - \mu(p_0) \in \ker A^T.$$

By A_{m-1} we have that the level sets of μ_A are connected. Let $p_1 \in \mu_A^{-1}(\xi)$ be another point in the same level set and let p_t be a path connecting p_0 to p_1. Since $\ker A^\top$ is 1-dimensional we have:

$$\begin{aligned}\mathrm{Conv}\{\mu(p_0), \mu(p_1)\} &= \{(1-s)\mu(p_0) + s\mu(p_1) \mid s \in I\} \\ &\subseteq \{\mu(p_t) \mid t \in I\} \\ &\subseteq \mu(M),\end{aligned}$$

where $I = [0, 1]$, is the unit interval.

Let now p_0 and p_1 be two points of M and let U_i be an open neighborhood of p_i (for $i = 0, 1$). It is always possible to find points $p^{U_i} \in U_i$ (for $i = 0, 1$) and an injective matrix $A \in \mathbb{Z}^{m \times (m-1)}$ such that $\mu(p_1^{U_1}) - \mu(p_0^{U_0}) \in \ker A^\top$. We can also take the limit over all the open neighbourhoods of p_0 and p_1:

$$\lim_{p_i \in U_i \in \mathrm{Op}(M)} p_i^{U_i} = p_i, \quad i = 0, 1.$$

We now notice that $(1+t)\mu(p_0^{U_0}) + t\mu(p_1^{U_1})$ lies in $\mu(M)$ for all $t \in I$. Since μ is continuous we have:

$$\lim_{\substack{p_i \in U_i \in \mathrm{Op}\, M \\ i=0,1}} (1+t)\mu(p_0^{U_0}) + t\mu(p_1^{U_1}) = (1+t)\mu(p_0) + t\mu(p_1)$$

and since $\mu(M)$ is compact (and therefore closed in $\mathbb{R}^m$):

$$(1+t)\mu(p_0) + t\mu(p_1) \in \mu(M)$$

thus $\mu(M)$ is convex.

Step 8 Finally we recall that $\operatorname{Fix}\psi$ is the finite union of connected symplectic submanifolds $F_1, \ldots, F_N$ and that μ is constant on each F_j: we denote $\eta_j = \mu(C_j)$, for $j = 1, \ldots, N$. By convexity of $\mu(M)$ we have:

$$\operatorname{Conv}\{\eta_1, \ldots, \eta_N\} \subset \mu(M).$$

To prove the other inclusion let $\xi \notin \operatorname{Conv}\{\eta_1, \ldots, \eta_N\}$ be a point outside of the convex hull of the image of the fixed points. Let $X \in \mathbb{R}^m$ be a vector with independent components over $\mathbb{Q}$ such that

$$\langle \xi, X \rangle > \left\langle \eta_j, X \right\rangle \quad \forall j = 1, \ldots, N.$$

We recall that, because the zeros of $X^\#$ correspond to the fixed points of ψ, μ^X achieves its maximum on $\operatorname{Fix}\psi$. So we have:

$$\langle \xi, X \rangle > \sup_{p \in M} \langle \mu(p), X \rangle .$$

which in turn proves that $\xi \notin \mu(M)$, hence the claim:

$$\mu(M) = \operatorname{Conv}\{\eta_1, \ldots, \eta_N\}$$

the image of the moment map is the convex hull of the image of its fixed points. This concludes the proof. ∎

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