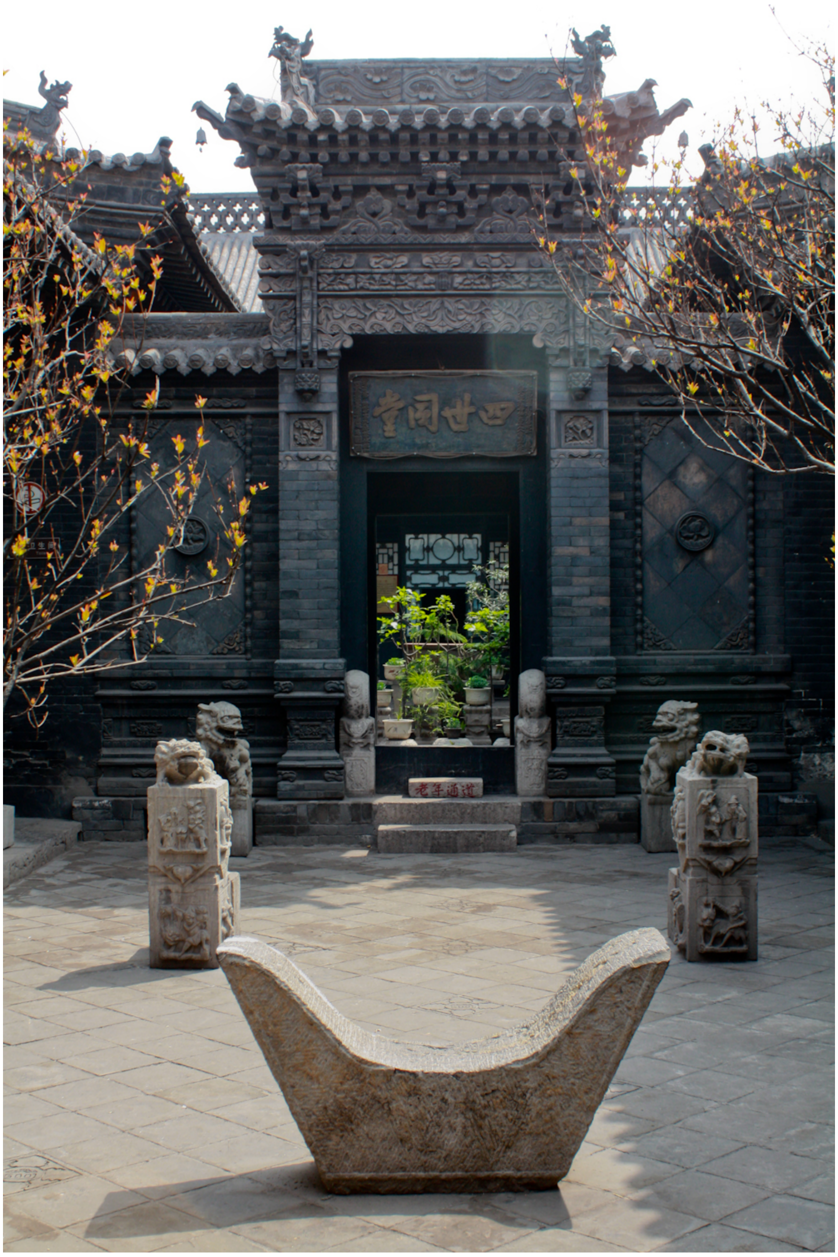
JOHN R. CALABRESE

ON A THEOREM BY BEĬLINSON



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ON A THEOREM BY BEĬLINSON

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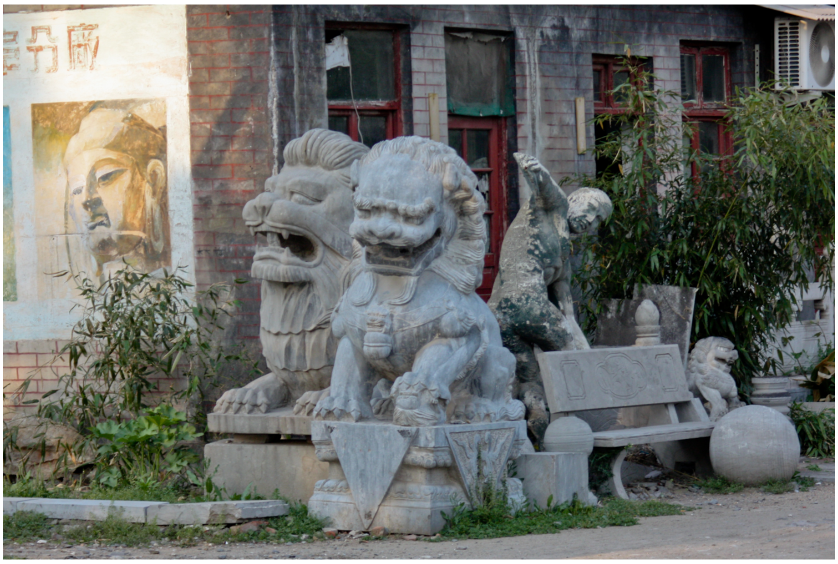

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To Pitú,
who takes care of me every day.

To Fede,
who puts up with me every day.

ABSTRACT

A classical theorem by Beĭlinson, regarding the bounded derived category of coherent sheaves on projective space, is proved. In the first chapter we collect some results from homological algebra. In the second chapter we apply these results in the context of Algebraic Geometry. In the third chapter we finally come to the main theorems. We prove three equivalences of $\mathbf{D}^{\mathrm{b}}(\mathsf{Coh}\mathbb{P}^n)$ with other, somewhat simpler, triangulated categories, introducing on the way the concept of tilting sheaves.

We suggest the reader to start with Chapter III, and use the first two chapters as a reference.

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There are of course many people that I'd like to thank: my family, my friends (from maths, physics, Palermo and capoeira), my present and past supervisors, Will and Ed for their help, my eternal housemate Enrico, my personal encyclopedia Giuseppe, my fellow student Giacomo and my henchmen Chiara and Daniele (without whom meeting the submission deadlines would have been literally impossible).

But most of all, I'd like to thank Fabio, who I consider my mathematical older brother. I will forever be in debt to his knowledge, his everlasting patience and his great passion.

London, September 2009 J.R.C.

CONTENTS

I

HOMOLOGICAL ALGEBRA - A QUICK TOUR

Contents

Here we give an introduction to triangulated and derived categories very much in the spirit of the first chapter of [KS90]. We omit some of the proofs and refer to the literature. To get a hold of the ideas behind these topics we suggest [Tho01], a survey which draws inspiration both from the topological and algebro-geometric point of view. For a truly extensive treatment, much beyond the scope of this thesis, one may consult [KS06]. Other references we keep in mind are [Huy06] (especially for the purpose of this thesis), the classic [Har66], [GM03] and the last chapter of [Wei94]. We assume familiarity with abelian categories and with the language of spectral sequences.

We deliberately disregard set-theoretical problems, in particular we never distinguish between small and big categories. A way to overcome these difficulties is to use arguments involving *universes*, as done for example in [KS06]. Quoting [GM03]:

> We will always assume, whenever necessary, that all required hygiene regulations are obeyed.

I.1 An Idea

Let us sketch the idea behind derived categories. Consider the following setting. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let

$$F : \mathcal{A} \to \mathcal{B}$$

be a left exact functor. Suppose moreover that $\mathcal{A}$ has enough injectives. One classically defines the *right derived functors* of F

$$R^i F : \mathcal{A} \to \mathcal{B}$$

as follows. For each object $X \in \mathcal{A}$ one picks an injective resolution $I^{\cdot}(X)$ of X, which is an exact sequence

$$0 \to X \to I^0(X) \to I^1(X) \to \cdots.$$

Then $R^iF(X)$ is defined to be the i-th cohomology of the complex

$$F(I^{\cdot}(X)) : \cdots \to 0 \to F(I^0(X)) \to F(I^1(X)) \to \cdots.$$

Similarly one defines *left* derived functors. One proves that, up to isomorphism, the definition of the derived functor does not depend on the resolutions we've chosen. Furthermore, to compute $R^iF(X)$ we could also have chosen a resolution $J^{\cdot}$, made up of F-*acyclic* objects (where acyclic means that all higher derived functors vanish).

As any object X of $\mathcal{A}$ can be treated as a 0-complex, viz. a complex

$$\cdots \to 0 \to X \to 0 \to \cdots$$

where X sits in degree zero, one notices that a *resolution* $I^{\cdot}$ of X is nothing but a *quasi-isomorphism* between X, regarded as a complex, and a complex $I^{\cdot}$.

Thus what one does to define derived functors is simply to identify an object X of $\mathcal{A}$ with a complex $I^{\cdot}$, quasi-isomorphic to it and made up of acyclic objects. What the derived category does is precisely this: it identifies any two quasi-isomorphic complexes, thereby identifying any object (seen as a 0-complex) with its resolutions.

There is also another reason to introduce derived categories. The functors R^iF are defined as the *cohomology* of some complex $F(I^{\cdot})$. When taking cohomology we lose information encoded in the original complex $F(I^{\cdot})$. Because of this, one wishes to redefine the derived functors. On the level of derived categories one defines the (total) right derived functor $\mathbf{R}F$ of F, which takes complexes in $\mathcal{A}$ and gives complexes in $\mathcal{B}$. The cohomology objects of $\mathbf{R}F$ are the previously denifed *classical* derived functors R^iF. Hence, when using the derived functor $\mathbf{R}F$, one ends up with complexes, which retain more information than their cohomologies. One might say (with thanks to F. Tonini and apologies to [Tho01]) that derived categories are constructed under the motto

> Cohomology: good.
> Complexes: way better.

Derived Categories

Before giving precise statements we sketch the definition of the derived category. One starts with a given abelian category $\mathcal{A}$, from which the category $\mathbf{C}(\mathcal{A})$ of complexes of $\mathcal{A}$ can be constructed. Let $X^{\cdot}, Y^{\cdot} \in \mathbf{C}(\mathcal{A})$ be two complexes and let

$$s : X^{\cdot} \to Y^{\cdot}$$

be a quasi-isomorphism. Although, by definition, s is an isomorphism on cohomology, an inverse s^{-1} needn't exist. In order to identify $X^{\cdot}$ and $Y^{\cdot}$ we put in by hand an inverse s^{-1}. Namely, if $\mathcal{S} = \mathsf{Qis}$ is the class of quasi-isomorphisms, what we want is to *localize* the category $\mathbf{C}(\mathcal{A})$ with respect to $\mathcal{S}$: we are looking for the *smallest* category $\mathbf{D}(\mathcal{A})$, containing $\mathbf{C}(\mathcal{A})$, in which all elements of $\mathcal{S}$ are invertible. There is a precise categorical notion for the construction above. If C is category and $\mathcal{S}$ is a class of morphisms then a *localization* of C by $\mathcal{S}$ is a category $\mathsf{C}_{\mathcal{S}}$ together with a functor

$$Q : \mathsf{C} \longrightarrow \mathsf{C}_{\mathcal{S}}$$

called the *quotient* functor, such that:

- For any $s \in \mathcal{S}$, $Q(s)$ is an isomorphism in $\mathsf{C}_{\mathcal{S}}$;
- For any other category D and any other functor

  $$F : \mathsf{C} \longrightarrow \mathsf{D}$$

  such that $F(s)$ is an isomorphism for any $s \in \mathcal{S}$, then F factors uniquely through Q. In other words there exists a unique functor

  $$F_{\mathcal{S}} : \mathsf{C}_{\mathcal{S}} \longrightarrow \mathsf{D}$$

  such that $F = F_{\mathcal{S}} \circ Q$.

Thus we define the *derived category* $\mathbf{D}(\mathcal{A})$ to be the localization of $\mathbf{C}(\mathcal{A})$ with respect to Qis, the class of quasi-isomorphisms.

Derived Functors

Let's turn to derived functors. Suppose we are given a functor

$$F : \mathcal{A} \longrightarrow \mathcal{B}$$

between two abelian categories. There is an obvious extension of F to the category of complexes:

$$\mathbf{C}(F) : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{B})$$

which we still denote by F. Let us write Q for both the quotient functors to the derived categories and consider

$$F' = Q \circ F : \mathbf{C}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}).$$

If F is an exact functor then it commutes with homology and thus sends quasi-isomorphisms into quasi-isomorphisms, namely: $F(s) \in \mathsf{Qis}$, for any $s \in \mathsf{Qis}$. Therefore, for any $s \in \mathsf{Qis}$, $F'(s) = QF(s)$ is an isomorphism. Hence F' admits a factorization, which we still denote by F,

$$F : \mathbf{D}(\mathcal{A}) \longrightarrow \mathbf{D}(\mathcal{B}),$$

viz. the following diagram can be completed to a commutative square.

$$\begin{tikzcd}
\mathbf{C}(\mathcal{A}) \arrow[r, "F"] \arrow[d, "Q"'] & \mathbf{C}(\mathcal{B}) \arrow[d, "Q"] \\
\mathbf{D}(\mathcal{A}) \arrow[r, dashed] & \mathbf{D}(\mathcal{B})
\end{tikzcd}$$

Suppose now we are given a functor F, which is only *left* (or right) exact. It is no longer true in general that F extends to the derived category as before. The right and left derived functors are an approximation of this desired extension. Their definition is of rather technical nature, but in most cases of interest it is easy to compute (from a theoretical point of view). One has in fact various results (the first of which is THEOREM I.6.3) which when applied to derived categories roughly amount to saying that: given the existence of a big enough subcategory of F-acyclic objects, the derived functor $\mathbf{R}F$ can be computed by resolving by acyclics and then applying F.

Structure of the Derived Category

Unfortunately the derived category of an abelian category is in general not itself abelian. It is however additive and it posses some additional structure: that of a *triangulated* category. In order to recover this last structure we break down the construction of the derived category into two steps. First: from $\mathcal{A}$ we build $\mathsf{C}(\mathcal{A})$, the category of complexes. Second: we define $\mathsf{K}(\mathcal{A})$ as the category whose objects are complexes of $\mathcal{A}$ and whose morphisms are morphisms of complexes modulo homotopy. It turns out that $\mathsf{K}(\mathcal{A})$ is a triangulated category, which is what gives this same structure on the derived category. Finally we define $\mathsf{D}(\mathcal{A})$ as the localization of $\mathsf{K}(\mathcal{A})$ with respect to quasi-isomorphisms. The two definitions of $\mathsf{D}(\mathcal{A})$ coincide (see [GM03]).

I.2 ABELIAN AND TRIANGULATED CATEGORIES

$\Bbbk$-linear Categories

Let's start with a category $\mathcal{A}$. We say that $\mathcal{A}$ is $\Bbbk$-*linear* (for a fixed commutative ring $\Bbbk$) if every Hom-set is endowed with the structure of a $\Bbbk$-module such that composition $\circ$ is bilinear. A $\Bbbk$-*linear functor* between two $\Bbbk$-linear categories $\mathcal{A}$ and $\mathcal{B}$ is a functor $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ such that the natural map

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{X}, \mathrm{Y}) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(\mathrm{FX}, \mathrm{FY})$$

is $\Bbbk$-linear for all objects $\mathrm{X}, \mathrm{Y} \in \mathcal{A}$. A full $\Bbbk$-linear subcategory is a full subcategory $\mathcal{B}$ such that the inclusion functor is $\Bbbk$-linear.

Additive Categories

A $\Bbbk$-*additive* category is a $\Bbbk$-linear category $\mathcal{A}$ with a zero object and satisfying

Ab1 For any two objects $\mathrm{X}, \mathrm{Y} \in \mathcal{A}$ there exists a third object (the *sum* of X and Y) $\mathrm{X} \oplus \mathrm{Y}$ and four morphisms

$$\mathrm{X} \xrightarrow{\iota_{\mathrm{X}}} \mathrm{X} \oplus \mathrm{Y} \xleftarrow{\iota_{\mathrm{Y}}} \mathrm{Y}$$
$$\mathrm{X} \xleftarrow{p_{\mathrm{X}}} \mathrm{X} \oplus \mathrm{Y} \xrightarrow{p_{\mathrm{Y}}} \mathrm{Y}$$

such that

$$\begin{aligned} p_{\mathrm{Y}}\iota_{\mathrm{X}} &= 0, \\ p_{\mathrm{X}}\iota_{\mathrm{Y}} &= 0, \\ p_{\mathrm{X}}\iota_{\mathrm{X}} &= \mathrm{Id}_{\mathrm{X}}, \\ p_{\mathrm{Y}}\iota_{\mathrm{Y}} &= \mathrm{Id}_{\mathrm{Y}}, \\ \iota_{\mathrm{X}}p_{\mathrm{X}} + \iota_{\mathrm{Y}}p_{\mathrm{Y}} &= \mathrm{Id}_{\mathrm{X}\oplus\mathrm{Y}}\,. \end{aligned} \tag{I.1}$$

One immediately notices the following.

I.2.1 LEMMA
Let $\mathcal{A}$, X and Y be as above. Then the squares

$$\begin{tikzcd} \mathrm{X} \oplus \mathrm{Y} \arrow[r, "p_{\mathrm{Y}}"] \arrow[d, "p_{\mathrm{X}}"'] & \mathrm{Y} \arrow[d] \\ \mathrm{X} \arrow[r] & 0 \end{tikzcd} \qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathrm{Y} \arrow[d, "\iota_{\mathrm{Y}}"] \\ \mathrm{X} \arrow[r, "\iota_{\mathrm{X}}"] & \mathrm{X} \oplus \mathrm{Y} \end{tikzcd}$$

are respectively cartesian *and* cocartesian. *Therefore* $(X \oplus Y, p_X, p_Y)$ *is the* product *of* X *and* Y *and* $(X \oplus Y, \iota_X, \iota_Y)$ *is the* coproduct *of* X *and* Y.

Products and Coproducts

We take a moment to establish some notation. In an arbitrary category, if $f : W \to X$ and $g : W \to Y$ are two morphisms, we denote by

$$f \times g = \begin{pmatrix} f \\ g \end{pmatrix} : W \longrightarrow X \times Y$$

the induced map from W to the product $X \times Y$ (assuming it exists). Dually, if $h : X \to Z$ and $k : Y \to Z$ are two arrows, we denote by

$$h \sqcup k = (h, k) : X \sqcup Y \longrightarrow Z$$

(and sometimes $h \oplus k$) the induced map from the coproduct (assuming it exists) to Z. Given four objects X_1, X_2 and Y_1, Y_2 and four morphisms $f_{ij} : X_i \to Y_j$, there are two ways to obtain a map $X_1 \sqcup X_2 \to Y_1 \times Y_2$. The first is

$$f^1 = (f_{11} \times f_{12}) \sqcup (f_{21} \times f_{22})$$

and the second is

$$f^2 = (f_{11} \sqcup f_{21}) \times (f_{12} \sqcup f_{22}).$$

Thanks to the universal properties of products and coproducts, the two coincide and we often employ the notation

$$f^1 = f^2 = \begin{pmatrix} f_{11} & f_{21} \\ f_{12} & f_{22} \end{pmatrix}.$$

From now on we drop the prefix $\Bbbk$- and speak of *linear* (or *pre-additive*) and *additive* (and later *abelian*) categories. It is useful to observe that if $\mathcal{A}$ is additive then the opposite category $\mathcal{A}^\circ$ is also additive. Also, if F is an additive functor, then $F(X \oplus Y) \simeq FX \oplus FY$.

Complexes

A *complex* in $\mathcal{A}$ is a sequence of objects $X^\cdot = (X^n)_{n \in \mathbb{Z}}$ with maps, called *differentials*, $(d_X^n : X^n \to X^{n+1})_n$ between them, represented by

$$X^\cdot : \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots,$$

such that the composition of any two consecutive maps is zero. A *morphism of complexes* is a collection of maps $(f^n : X^n \to Y^n)_n$

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that

$$d_Y^n f^n = f^{n+1} d_X^n$$

for all $n \in \mathbb{Z}$. Thus complexes form a category $\mathbf{C}(\mathcal{A})$, and it is easy to check that it is additive. For example, if $X^\cdot$, $Y^\cdot$ are two complexes then their sum

Bounded Complexes

is given by the complex $(X^n \oplus Y^n)_n$, with differentials $(d^n_X \oplus d^n_Y)_n$. We may also consider the full additive subcategories $\mathbf{C}^*(\mathcal{A})$, where $* = \mathsf{ub}, +, -, \mathsf{b}$:

$$\begin{aligned} \mathrm{Ob}\,\mathbf{C}^{\mathsf{ub}}(\mathcal{A}) &= \mathbf{C}(\mathcal{A}) \text{ (unbounded complexes)}, \\ \mathrm{Ob}\,\mathbf{C}^{+}(\mathcal{A}) &= \{X^\cdot \mid X^n = 0 \text{ for } n \ll 0\} \text{ (bounded below)}, \\ \mathrm{Ob}\,\mathbf{C}^{-}(\mathcal{A}) &= \{X^\cdot \mid X^n = 0 \text{ for } n \gg 0\} \text{ (bounded above)}, \\ \mathrm{Ob}\,\mathbf{C}^{\mathsf{b}}(\mathcal{A}) &= \{X^\cdot \mid X^n = 0 \text{ for } |n| \gg 0\} \text{ (bounded).} \end{aligned}$$

We often omit to write Ob, simply using $X \in \mathcal{A}$ to mean that X is an object of the category $\mathcal{A}$. If $F : \mathcal{A} \to \mathcal{B}$ is additive, there is an obvious extension of F to an additive functor $\mathbf{C}^*(F) : \mathbf{C}^*(\mathcal{A}) \to \mathbf{C}^*(\mathcal{B})$, for $* = \mathsf{ub}, +, -, \mathsf{b}$, which we sometimes denote simply by F.

Homotopy

A morphism of complexes f is *null homotopic* if there exists a collection of maps $(s^n : X^n \to Y^{n-1})_n$

$$\begin{tikzcd} \cdots \arrow[r] & X^{n-1} \arrow[r, "d_X^{n-1}"] & X^n \arrow[r, "d_X^n"] \arrow[dl, "s^n"'] & X^{n+1} \arrow[r] \arrow[dl, "s^{n+1}"'] & \cdots \\ \cdots \arrow[r] & Y^{n-1} \arrow[r, "d_Y^{n-1}"] & Y^n \arrow[r, "d_Y^n"] & Y^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

such that

$$d_Y^{n-1}s^n + s^{n+1}d_X^n = f^n$$

for all $n \in \mathbb{Z}$. Two morphisms f, g are *homotopic*, and we write $f \sim g$, if $f - g$ is null homotopic. We denote by $\mathrm{Ht}(X^\cdot, Y^\cdot)$ the submodule of $\mathrm{Hom}(X^\cdot, Y^\cdot)$ consisting of null homotopic maps. Homotopy is compatible with composition, in the sense that if $f \sim f'$ and $g \sim g'$ then $gf \sim g'f'$ (when composition makes sense). Thus we may form the category $\mathbf{K}(\mathcal{A})$, along with the subcategories $\mathbf{K}^*(\mathcal{A})$, as follows.

$$\begin{aligned} \mathrm{Ob}\,\mathbf{K}(\mathcal{A}) &= \mathrm{Ob}\,\mathbf{C}(\mathcal{A}) \\ \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X^\cdot, Y^\cdot) &= \frac{\mathrm{Hom}_{\mathbf{C}(\mathcal{A})}(X^\cdot, Y^\cdot)}{\mathrm{Ht}(X^\cdot, Y^\cdot)}. \end{aligned}$$

The categories $\mathbf{K}^*(\mathcal{A})$ are also additive. We say that two complexes $X^\cdot$ and $Y^\cdot$ are *homotopy equivalent* if they are isomorphic in $\mathbf{K}$, viz. there exist two morphisms in $\mathbf{C}$, $f : X^\cdot \to Y^\cdot$ and $g : Y^\cdot \to X^\cdot$, such that $gf \sim \mathrm{Id}_{X^\cdot}$ and $fg \sim \mathrm{Id}_{Y^\cdot}$. One also notices that if $F : \mathcal{A} \to \mathcal{B}$ is additive and f is null homotopic, then $F(f)$ is null homotopic. Therefore, the extensions $\mathbf{C}^*(F)$ pass on to the homotopy category yielding functors $\mathbf{K}^*(F)$, for $* = \mathsf{ub}, +, -, \mathsf{b}$, which again we sometimes denote simply by F. , then if f is null homotopic

Shifts

Let $k \in \mathbb{Z}$ be an integer, we define the *shift by k* of a complex $X^\cdot$ as the complex

$$(X^\cdot[k]^n = X^{k+n})_n,$$

with differential

$$(d^n_{X^\cdot[k]} = (-1)^k d^{n+k}_{X^\cdot})_n.$$

We can also shift morphisms with the rule: $(f[k]^n = f^{k+n})_n$. Hence the shift (also called *translation*) $[k]$ is an additive automorphism of $\mathbf{C}^*(\mathcal{A})$ and passes on to $\mathbf{K}^*(\mathcal{A})$, for $* = \mathrm{ub}, +, -, \mathrm{b}$. We also notice that for any two integers $h, k \in \mathbb{Z}$ we have $[h+k] = [h] \circ [k]$. We sometimes use T to denote the shift by one. We now pause our discussion of complexes to introduce abelian categories.

Abelian Categories

Let now $\mathcal{A}$ be an additive category. We define the *kernel* $\operatorname{Ker} f$ of a morphism $f : \mathrm{X} \to \mathrm{Y}$ to be the equalizer of the two parallel arrows

$$f, 0 : \mathrm{X} \rightrightarrows \mathrm{Y}.$$

Equivalently the kernel might be defined as an object K together with a morphism $\mathrm{K} \to \mathrm{X}$, satisfying the universal property visualized by the following commutative diagram.

$$\begin{tikzcd}
& \mathrm{X} \arrow[r, "f"] & \mathrm{Y} \\
& \mathrm{K} \arrow[u] \arrow[r] & 0 \arrow[u] \\
\mathrm{W} \arrow[uur] \arrow[ur, dashed] \arrow[urr] & &
\end{tikzcd}$$

where the dashed arrow stands for *exists unique*. Dually one defines the *cokernel* $\operatorname{Coker} f$ of f as the coequalizer of the two parallel arrows

$$f, 0 : \mathrm{X} \rightrightarrows \mathrm{Y}.$$

Dually, the cokernel may be defined as a kernel in the opposite category, thus satisfying the universal property expressed by the opposite of the diagram above. We notice that $\operatorname{Ker} f \simeq \operatorname{Ker}(-f)$ and $\operatorname{Coker} f \simeq \operatorname{Coker}(-f)$, for all morphisms f. As a consequence of the definitions the map $\operatorname{Ker} f \hookrightarrow \mathrm{X}$ is a monomorphism and $\mathrm{Y} \twoheadrightarrow \operatorname{Coker} f$ is an epimorphism. We then define the *image* $\operatorname{Im} f$ of f as $\operatorname{Ker}(\mathrm{Y} \twoheadrightarrow \operatorname{Coker} f)$ and the *coimage* $\operatorname{Coim} f$ of f as $\operatorname{Coker}(\operatorname{Ker} f \hookrightarrow \mathrm{X})$. There is a natural map (when the objects involved exist)

$$\begin{tikzcd}
\operatorname{Ker} f \arrow[r, hook] & \mathrm{X} \arrow[r, "f"] \arrow[d, two heads] & \mathrm{Y} \arrow[r, two heads] & \operatorname{Coker} f \\
& \operatorname{Coim} f \arrow[r, dashed] & \operatorname{Im} f \arrow[u, hook] &
\end{tikzcd}$$

constructed as follows. The morphism $\mathrm{X} \to \mathrm{Y} \to \operatorname{Coker} f$ is zero, hence it must factor through $\operatorname{Im} f$. In turn the map $\operatorname{Ker} f \to \mathrm{X} \to \operatorname{Im} f$ is zero so we obtain an arrow $\operatorname{Coim} f \to \operatorname{Im} f$. Dually one obtains another arrow $\operatorname{Coim} f \to \operatorname{Im} f$ by observing that $\operatorname{Ker} f \to \mathrm{X} \to \mathrm{Y}$ is zero. The two maps coincide as a consequence of the universal properties satisfied by kernels and cokernels.

We say that $\mathcal{A}$ is *abelian* if it satisfies

Ab2 For any morphism f, the kernel and cokernel of f exist. Moreover the natural map $\mathrm{Coim}\, f \to \mathrm{Im}\, f$ is an isomorphism.

We remark that if $\mathcal{A}$ is abelian then $\mathcal{A}^\circ$ is also abelian.

Let $\mathcal{A}$ be an abelian category. As a consequence of Ab2 we have that a morphism f in $\mathcal{A}$ is: a monomorphism if and only $\mathrm{Ker}\, f = 0$; it is an epimorphism if and only if $\mathrm{Coker}\, f = 0$; it is an isomorphism if and only if $\mathrm{Ker}\, f = 0 = \mathrm{Coker}\, f$.

Cohomology

To define the cohomology of a complex the following lemma is useful.

I.2.2 Lemma

Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms.

- *If g is a monomorphism then* $\mathrm{Ker}\, gf \sim \mathrm{Ker}\, f$*. Dually if f is an epimorphism then* $\mathrm{Coker}\, gf \sim \mathrm{Coker}\, g$*.*

- *If* $W \hookrightarrow X$ *is a monomorphism such that* $W \hookrightarrow X \to Y$ *is zero, then the induced map* $W \hookrightarrow \mathrm{Ker}\, f$ *is also a monomorphism. Dually if* $Y \twoheadrightarrow Z$ *is an epimorphism such that* $X \to Y \twoheadrightarrow Z$ *is zero, then the induced map* $\mathrm{Coker}\, f \twoheadrightarrow Z$ *is also an epimorphism.*

Given two consecutive morphisms $f : X \to Y$ and $g : Y \to Z$ such that $gf = 0$ we say that $X \to Y \to Z$ is a *complex* and we identify it with the sequence

$$\cdots \to 0 \to X \to Y \to Z \to 0 \to \cdots$$

where Y sits in degree zero, which is a complex in the sense of our earlier definition.

Given a complex

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we form the diagram

$$\begin{tikzcd}
 & \mathrm{Im}\, f \arrow[dr, hook] & & \mathrm{Ker}\, g \arrow[dl, hook'] & \\
X \arrow[ur, two heads] \arrow[rr, "f"] & & Y \arrow[rr, "g"] \arrow[dl, two heads] \arrow[dr, two heads] & & Z \\
 & \mathrm{Coker}\, f & & \mathrm{Coim}\, g \arrow[ur, hook] &
\end{tikzcd}$$

Since gf is zero, $\mathrm{Im}\, f \to Y \to Z$ is zero and so $\mathrm{Im}\, f \to Y$ factors through $\mathrm{Ker}\, g$, yielding the monomorphism $\varphi : \mathrm{Im}\, f \hookrightarrow \mathrm{Ker}\, g$. Dually we obtain a factorization of $Y \to \mathrm{Coim}\, g$ through an epimorphism $\psi : Y \twoheadrightarrow \mathrm{Coim}\, g$.

$$\begin{tikzcd}
& \mathrm{Im}\, f \arrow[rr, hook, "\varphi"] \arrow[dr, hook] & & \mathrm{Ker}\, g \arrow[r, two heads] \arrow[dl, hook] & \mathrm{Coker}\,\varphi \\
\mathrm{X} \arrow[ur, two heads] \arrow[rr, "f"] & & \mathrm{Y} \arrow[rr, "g"] \arrow[dl, two heads] \arrow[dr, two heads] & & \mathrm{Z} \\
\mathrm{Ker}\,\psi \arrow[r, hook] & \mathrm{Coker}\, f \arrow[rr, two heads, "\psi"] & & \mathrm{Coim}\, g \arrow[ur, hook] &
\end{tikzcd}$$

We want to show that $\mathrm{Ker}\,\psi \simeq \mathrm{Coker}\,\varphi$. First we notice that

$$\mathrm{Coker}\,\varphi = \mathrm{Coker}(\mathrm{Im}\, f \to \mathrm{Ker}\, g) \simeq \mathrm{Coker}(\mathrm{X} \to \mathrm{Ker}\, g)$$

and

$$\mathrm{Ker}\,\psi = \mathrm{Ker}(\mathrm{Coker}\, f \to \mathrm{Coim}\, g) \simeq \mathrm{Ker}(\mathrm{Coker}\, f \to \mathrm{Z}).$$

Let now u be the composition $\mathrm{Ker}\, g \to \mathrm{Y} \to \mathrm{Coker}\, f$. Since $\mathrm{Im}\, f \to \mathrm{Y} \to \mathrm{Coker}\, f$ is zero we obtain a morphism $\mathrm{Im} \to \mathrm{Ker}\, u$. And since $\mathrm{Ker}\, u \to \mathrm{Coker}\, f$ is zero we obtain another morphism $\mathrm{Ker}\, u \to \mathrm{Im}\, f$. From the universal properties of kernels it follows that $\mathrm{Im} \to \mathrm{Ker}\, u \to \mathrm{Im}\, f$ is the identity and also $\mathrm{Ker}\, u \to \mathrm{Im}\, f \to \mathrm{Ker}\, u$ is the identity. Thus we have an isomorphism $\mathrm{Ker}\, u \simeq \mathrm{Im}\, f$. Dually we obtain an isomorphism $\mathrm{Coim}\, g \simeq \mathrm{Coker}\, u$. Hence

$$\begin{aligned}
\mathrm{Coker}(\mathrm{X} \to \mathrm{Ker}\, g) &\simeq \mathrm{Coker}\,\varphi \\
&= \mathrm{Coker}(\mathrm{Im}\, f \to \mathrm{Ker}\, g) \\
&\simeq \mathrm{Coker}(\mathrm{Ker}\, u \to \mathrm{Ker}\, g) \\
&\simeq \mathrm{Coim}\, u \\
&\simeq \mathrm{Im}\, u \\
&\simeq \mathrm{Ker}(\mathrm{Coker}\, f \to \mathrm{Coker}\, u) \\
&\simeq \mathrm{Ker}(\mathrm{Coker}\, f \to \mathrm{Coim}\, g) \\
&= \mathrm{Ker}\,\psi \\
&\simeq \mathrm{Ker}(\mathrm{Coker}\, f \to \mathrm{Z}).
\end{aligned} \tag{I.2}$$

I.2.3 Definition

Let $\mathrm{X} \to \mathrm{Y} \to \mathrm{Z}$ *be a complex as above.*

- *We define its* cohomology

$$\mathrm{H}(\mathrm{X} \to \mathrm{Y} \to \mathrm{Z})$$

  to be one of the isomorphic objects in (I.2). *For example*

$$\mathrm{H}(\mathrm{X} \to \mathrm{Y} \to \mathrm{Z}) = \mathrm{Coker}(\mathrm{Im}\, f \to \mathrm{Ker}\, g)$$

  to fix our ideas.

- *We say that it is an* exact sequence *if its cohomology vanishes*

$$\mathrm{H}(\mathrm{X} \to \mathrm{Y} \to \mathrm{Z}) = 0.$$

Given a long complex

$$X^{\cdot} : \cdots \to X^{n-1} \to X^{n} \to X^{n+1} \to \cdots$$

we define its n-th cohomology $H^n(X^{\cdot})$ *to be*

$$H(X^{n-1} \to X^{n} \to X^{n+1}),$$

and we say that $X^{\cdot}$ *is a* long exact sequence *or an* acyclic complex *if all its coholomogies vanish, viz.* $H^n(X^{\cdot}) = 0$, *for all $n \in \mathbb{Z}$.*

Notice that $H^n(X^{\cdot}[k]) = H^{n+k}(X^{\cdot})$. We also notice that given complex $X \to Y \to Z$, the cohomology of the opposite complex in the opposite category is the opposite of the cohomology of the original complex, namely

$$H(Z^{\circ} \to Y^{\circ} \to X^{\circ}) = (H(X \to Y \to Z))^{\circ}.$$

Exact Sequences

A sequence $0 \to X \to Y$ is exact if and only if $X \to Y$ is a monomorphism; a sequence $X \to Y \to 0$ is exact if and only if $X \to Y$ is an epimorphism. A complex

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

is exact if and only if f is a monomorphism, g is an epimorphism and $\operatorname{Ker} g \simeq \operatorname{Im} f$. In such cases we speak of *short exact sequences*. Any morphism $f : X \to Y$ may be decomposed into short exact sequences.

$$0 \to \operatorname{Ker} f \to X \to \operatorname{Coim} f \to 0$$
$$0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0.$$

An additive functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories is:

- *left exact* if for any exact sequence

  $$0 \to X \to Y \to Z$$

  the sequence

  $$0 \to FX \to FY \to FZ$$

  is exact;

- *right exact* if for any exact sequence

  $$X \to Y \to Z \to 0$$

  the sequence

  $$FX \to FY \to FZ \to 0$$

  is exact;

- *exact* if it is both left and right exact, or equivalently if for any exact sequence

  $$0 \to X \to Y \to Z \to 0$$

  the sequence

  $$0 \to FX \to FY \to FZ \to 0$$

  is exact.

I.2.4 Remark Given an object X of a $\Bbbk$-abelian category $\mathcal{A}$, the functors

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{X}, -) : \mathcal{A} \to \Bbbk\text{-Mod}$$
$$\mathrm{Hom}_{\mathcal{A}}(-, \mathrm{X}) : \mathcal{A}^{\circ} \to \Bbbk\text{-Mod}$$

are both left exact.

The Embedding Theorem

Our main example of an abelian category is R-Mod, the category of left modules over a ring R with linear maps as arrows. Many of the proofs in R-Mod often involve the chasing of elements in some diagram. In order to be able to *diagram chase* in an arbitrary abelian category $\mathcal{A}$, one may proceed in two ways. On one hand it is possible to define formal *elements* in $\mathcal{A}$, as done for example in [ML98, CHAPTER VIII], from which one deduces some diagram chasing lemmas. On the other hand one can use the Freyd-Mitchell embedding theorem which we now state (for a proof see [KS06, THEOREM 9.6.10]).

I.2.5 THEOREM (FREYD-MITCHELL)
Let $\mathcal{A}$ be a small[1] *abelian category. There exist a ring* R *and an exact fully faithful functor* $\mathcal{A} \to$ R-Mod.

A typical application of the theorem goes likes this: one starts from a finite diagram in an abelian category, then one takes the full abelian subcategory containing all objects involved in the diagram and uses the embedding theorem to prove the desired result by diagram chase in R-Mod, finally one pulls the result back to the original category. As a consequence one has the *five lemma* and the *snake lemma* in any arbitrary abelian category.

I.2.6 THEOREM (FIVE LEMMA)
Consider a commutative diagram whose rows are complexes

$$\begin{tikzcd}
\mathrm{X}_0 \arrow[r] \arrow[d, "f_0"'] & \mathrm{X}_1 \arrow[r] \arrow[d, "f_1"'] & \mathrm{X}_2 \arrow[r] \arrow[d, "f_2"'] & \mathrm{X}_3 \arrow[d, "f_3"'] \\
\mathrm{Y}_0 \arrow[r] & \mathrm{Y}_1 \arrow[r] & \mathrm{Y}_2 \arrow[r] & \mathrm{Y}_3
\end{tikzcd}$$

where $\mathrm{X}_0 \to \mathrm{X}_1 \to \mathrm{X}_2$ *and* $\mathrm{Y}_0 \to \mathrm{Y}_1 \to \mathrm{Y}_2$ *are exact sequences. Then*

- *If f_0 is a epimorphism and f_1, f_3 are monomorphisms, then f_2 is a monomorphism.*
- *If f_3 is a monomorphism and f_0, f_2 are epimorphisms, then f_1 is an epimorphism.*

The *classical* five lemma is consequence of the above theorem and of the fact that in an abelian category an arrow that is both a monomorphism and an epimorphism is an isomorphism. It asserts that given a commutative diagram with exact rows

1 A *small* category is a category such that the collection of all arrows forms a set belonging to some universe $\mathcal{U}$, fixed beforehand.

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"'] & X_1 \arrow[r] \arrow[d, "f_1"'] & X_2 \arrow[r] \arrow[d, "f_2"'] & X_3 \arrow[r] \arrow[d, "f_3"'] & X_4 \arrow[d, "f_4"'] \\
Y_0 \arrow[r] & Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & Y_4
\end{tikzcd}$$

if f_0, f_1, f_3 and f_4 are isomorphisms then f_2 is an isomorphism also.

I.2.7 Theorem (Snake Lemma)
A commutative diagram with exact rows

$$\begin{tikzcd}
 & X' \arrow[r, "f"] \arrow[d, "u"'] & X \arrow[r, "g"] \arrow[d, "v"'] & X'' \arrow[r] \arrow[d, "v"'] & 0 \\
0 \arrow[r] & Y' \arrow[r, "h"] & Y \arrow[r, "k"] & Y'' &
\end{tikzcd}$$

gives rise to an exact sequence

$$\operatorname{Ker} u \xrightarrow{f_1} \operatorname{Ker} v \xrightarrow{g_1} \operatorname{Ker} w \xrightarrow{\varphi} \operatorname{Coker} u \xrightarrow{h_1} \operatorname{Coker} v \xrightarrow{k_1} \operatorname{Coker} w.$$

Full Subcategories

Let now $\mathcal{J}$ be a full subcategory of $\mathcal{A}$. We say that $\mathcal{J}$ is

- *closed by subobjects (quotients)* if for any monomorphism $W \hookrightarrow X$ (epimorphism $X \twoheadrightarrow Z$) with $X \in \mathcal{J}$ then $W \in \mathcal{J}$ ($Z \in \mathcal{J}$);
- *closed by kernels (cokernels)* if for any arrow $X \xrightarrow{f} Y$ in $\mathcal{J}$ we have $\operatorname{Ker} f \in \mathcal{J}$ ($\operatorname{Coker} f \in \mathcal{J}$);
- *closed by extensions* if for any short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X', X'' \in \mathcal{J}$ then $X \in \mathcal{J}$;
- *thick* if it is closed under kernels, cokernels and extensions;
- *generating (cogenerating)* if any $X \in \mathcal{A}$ is a quotient (subobject) of an object of $\mathcal{J}$;
- a *fully abelian subcategory* if it is abelian and the inclusion functor is exact.

Long Exact Sequence

Let us now return to complexes. Given an abelian category $\mathcal{A}$ the category $\mathbf{C}(\mathcal{A})$ is also abelian and $\mathbf{C}^{\mathrm{b}}(\mathcal{A})$ is thick in both $\mathbf{C}^{+}(\mathcal{A})$ and $\mathbf{C}^{-}(\mathcal{A})$, which in turn are thick in $\mathbf{C}(\mathcal{A})$. The abelian structure on $\mathbf{C}$ is given component-wise. For example the kernel of a morphism f is given by the complex $(\operatorname{Ker} f^n)_n$, with differentials the natural maps among the kernels.

If $X^{\cdot}$ is a complex the following is an exact sequence for all $n \in Z$:

$$0 \to H^n(X^{\cdot}) \to \operatorname{Coker} d_{X^{\cdot}}^{n-1} \xrightarrow{d_{X^{\cdot}}^{n}} \operatorname{Ker} d_{X^{\cdot}}^{n+1} \to H^{n+1}(X^{\cdot}) \to 0. \tag{I.3}$$

If $f : X^{\cdot} \to Y^{\cdot}$ is a map of complexes there is an induced map

$$H^n(f) : H^n(X^{\cdot}) \longrightarrow H^n(Y^{\cdot}).$$

One can then define the functors $H^n : \mathbf{C}(\mathcal{A}) \to \mathcal{A}$, for all $n \in \mathbb{Z}$.

I.2.8 Theorem
Any short exact sequence of complexes

$$0 \to X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to 0$$

gives rise to a long exact sequence

$$\cdots \to H^{n-1}(Z^{\cdot}) \to H^n(X^{\cdot}) \to H^n(Y^{\cdot}) \to H^n(Z^{\cdot}) \to H^{n+1}(X^{\cdot}) \to \cdots$$

which is natural, in the sense that given a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & X^{\cdot} \arrow[r] \arrow[d] & Y^{\cdot} \arrow[r] \arrow[d] & Z^{\cdot} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X'^{\cdot} \arrow[r] & Y'^{\cdot} \arrow[r] & Z'^{\cdot} \arrow[r] & 0 \end{tikzcd}$$

the square below commutes, for all $n \in \mathbb{Z}$.

$$\begin{tikzcd} H^n(Z^{\cdot}) \arrow[r] \arrow[d] & H^{n+1}(X^{\cdot}) \arrow[d] \\ H^n(Z'^{\cdot}) \arrow[r] & H^{n+1}(X'^{\cdot}) \end{tikzcd}$$

Proof
From the exact sequence we obtain commutative diagrams with exact rows

$$\begin{tikzcd} & \operatorname{Coker} d^{n-1}_{X^{\cdot}} \arrow[r] \arrow[d] & \operatorname{Coker} d^{n-1}_{Y^{\cdot}} \arrow[r] \arrow[d] & \operatorname{Coker} d^{n-1}_{Z^{\cdot}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \operatorname{Ker} d^{n+1}_{X^{\cdot}} \arrow[r] & \operatorname{Ker} d^{n+1}_{Y^{\cdot}} \arrow[r] & \operatorname{Ker} d^{n+1}_{Z^{\cdot}} & \end{tikzcd}$$

Applying the snake lemma plus the exact sequence (I.3) we obtain the desired long exact sequence. ❧

One can group all the cohomology functors H^n together into a single functor

$$H^{\cdot} : \mathbf{C}(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$$

where, given a complex $X^{\cdot}$, $H^{\cdot}(X^{\cdot})$ is the complex $(H^n(X^{\cdot}))_n$ with the zero maps for differentials. If f is null homotopic one notices that $H^{\cdot}(f) = 0$, therefore the functor $H^{\cdot}$ passes onto the homotopy category

$$H^{\cdot} : \mathbf{K}(\mathcal{A}) \longrightarrow \mathbf{C}(\mathcal{A}).$$

Of course $H^{\cdot}$ restricts to all the subcategories $\mathbf{C}^*$, $\mathbf{K}^*$, for $* = +, -, b$.

Quasi-isomorphisms

If we consider again a map f in $\mathbf{C}$, we say that f is a *quasi-isomorphism* (qis for short) if $H^{\cdot}(f)$ is an isomorphism, viz. $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$. The same definition goes for morphisms in $\mathbf{K}$.

We've already remarked that if we start from an abelian category $\mathcal{A}$, the categories $\mathsf{C}^*(\mathcal{A})$ are abelian. Unfortunately the categories $\mathsf{K}^*(\mathcal{A})$ are in general not abelian. They do have some structure though: they are an example of triangulated categories. To illustrate this structure we must introduce the mapping cone.

Fix again an additive category $\mathcal{A}$.

I.2.9 Definition

Let

$$f : X^{\cdot} \longrightarrow Y^{\cdot}$$

be a morphism of complexes. The mapping cone *of f is the complex* $\mathrm{Mc}(f)$ *defined as* $\mathrm{Mc}(f) = X^{\cdot}[1] \oplus Y^{\cdot}$ *with differential*

$$d_{\mathrm{Mc}(f)} = \begin{pmatrix} d_{X[1]} & 0 \\ f[1] & d_Y \end{pmatrix}.$$

Note that if $f \sim g$ then $\mathrm{Mc}(f) \simeq \mathrm{Mc}(g)$ in K. We have two natural maps

$$\alpha(f) : Y^{\cdot} \longrightarrow \mathrm{Mc}(f),$$

which is simply the inclusion of $Y^{\cdot}$ in the second factor of $\mathrm{Mc}(f)$ and

$$\beta(f) : \mathrm{Mc}(f) \longrightarrow X^{\cdot}[1]$$

which is the projection on the first factor. Before we continue we give a general definition.

Triangles

A *category with translation* is a pair (C, T) where C is a category and T is an automorphism of C, called the *translation* (or *shift*[2]). We sometimes use the notation $[1]$ for T when no confusion is possible. A *functor of categories with translation* is a functor $F : (\mathsf{C}, T) \to (\mathsf{C}', T')$ between the underlying categories such that $F \circ T = T' \circ F$. We define an *additive category with translation* to be a category with translation $(\mathcal{A}, T)$, where $\mathcal{A}$ is additive and T is also additive. Of course a *functor of additive categories with translation* is simply a functor of categories with translations which also is additive. Given an additive category $\mathcal{A}$, the categories $\mathsf{C}^*(\mathcal{A})$ and $\mathsf{K}^*(\mathcal{A})$, for $* = \mathsf{ub}, +, -, \mathsf{b}$, are all additive categories with translation, where the translation is given by the shift of complexes by one.

A *triangle* in an additive category with translation $(\mathcal{A}, T)$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX.$$

We sometimes write

$$X \xrightarrow{[1]} Y$$

to indicate a map from X to TY. Because of this triangles are also denoted by

$$X \longrightarrow Y \longrightarrow Z \xrightarrow{[1]} .$$

A *morphism of triangles* is a commutative diagram

2 Also called *suspension* by the topologists.

$$\begin{tikzcd}
X^\cdot \arrow[r] \arrow[d, "\phi"'] & Y^\cdot \arrow[r] \arrow[d] & Z^\cdot \arrow[r] \arrow[d] & X^\cdot[1] \arrow[d, "\phi[1]"'] \\
X'^\cdot \arrow[r] & Y'^\cdot \arrow[r] & Z'^\cdot \arrow[r] & X'^\cdot[1]
\end{tikzcd}$$

I.2.10 Lemma

Let $\mathcal{A}$ be an additive category and let $f : X^\cdot \to Y^\cdot$ be a morphism in $\mathsf{C}(\mathcal{A})$. Then there exists $\phi : X^\cdot[1] \to \mathrm{Mc}(\alpha(f))$ such that ϕ is an isomorphism in $\mathsf{K}(\mathcal{A})$ and such that the following diagram commutes in $\mathsf{K}(\mathcal{A})$.

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, dash, "\mathrm{Id}_{X^\cdot}"'] & Y^\cdot \arrow[r, "\alpha(f)"] \arrow[d, dash, "\mathrm{Id}_{Y^\cdot}"'] & \mathrm{Mc}(f)^\cdot \arrow[r, "\beta(f)"] \arrow[d, dash, "\mathrm{Id}_{\mathrm{Mc}(f)}"'] & X^\cdot[1] \arrow[r, "-f[1]"] \arrow[d, "\phi"'] & Y^\cdot[1] \arrow[d, dash, "\mathrm{Id}_{Y^\cdot[1]}"'] \\
X^\cdot \arrow[r, "f"] & Y^\cdot \arrow[r, "\alpha(f)"] & \mathrm{Mc}(f) \arrow[r, "\alpha(\alpha(f))"] & \mathrm{Mc}(\alpha(f)) \arrow[r, "\beta(\alpha(f))"] & Y^\cdot[1]
\end{tikzcd}$$

Proof

See [KS90, Lemma 1.4.2].

We say that a triangle $X^\cdot \to Y^\cdot \to Z^\cdot \to X^\cdot[1]$ is *distinguished* if it is isomorphic to a mapping cone triangle $X'^\cdot \to Y'^\cdot \to \mathrm{Mc}(X'^\cdot \to Y'^\cdot) \to X'^\cdot[1]$ as above.

I.2.11 Theorem

Let $\mathcal{A}$ be an additive category. Then the category $\mathsf{K}(\mathcal{A})$ together with the collection of distinguished triangles (d.t. for short) satisfies the following properties.

TR0 *A triangle isomorphic to a d.t. is a d.t.*

TR1 *The triangle*

$$X^\cdot \xrightarrow{\mathrm{Id}_{X^\cdot}} X^\cdot \longrightarrow 0 \longrightarrow X^\cdot[1]$$

is a d.t.

TR2 *Any $f : X^\cdot \to Y^\cdot$ can be embedded in a d.t.*

$$X^\cdot \xrightarrow{f} Y^\cdot \longrightarrow Z^\cdot \longrightarrow X^\cdot[1]$$

TR3 *A triangle*

$$X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{g} Z^\cdot \xrightarrow{h} X^\cdot[1]$$

is a d.t. if and only if

$$X^\cdot \xrightarrow{-f} Y^\cdot \xrightarrow{-g} Z^\cdot \xrightarrow{-h} X^\cdot[1]$$

is a d.t.

TR4 *Given*

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, "\alpha"'] & Y^\cdot \arrow[r, "g"] \arrow[d, "\beta"'] & Z^\cdot \arrow[r, "h"] & X^\cdot[1] \\
X'^\cdot \arrow[r, "f'"] & Y'^\cdot \arrow[r, "g'"] & Z'^\cdot \arrow[r, "h'"] & X'^\cdot[1]
\end{tikzcd}$$

where the top and bottom rows are distinguished triangles, and the square on the left is commutative there exists a (not necessarily unique) $\gamma : Z^\cdot \to Z'^\cdot$ giving rise to a morphism of distinguished triangles.

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, "\alpha"'] & Y^\cdot \arrow[r, "g"] \arrow[d, "\beta"'] & Z^\cdot \arrow[r, "h"] \arrow[d, dashed, "\gamma"'] & X^\cdot[1] \arrow[d, "\alpha[1]"'] \\
X'^\cdot \arrow[r, "f'"] & Y'^\cdot \arrow[r, "g'"] & Z'^\cdot \arrow[r, "h'"] & X'^\cdot[1]
\end{tikzcd}$$

TR5 *Given three d.t.s*

$$X^\cdot \xrightarrow{f} Y^\cdot \xrightarrow{h} Z'^\cdot \longrightarrow X^\cdot[1],$$
$$Y^\cdot \xrightarrow{g} Z^\cdot \xrightarrow{k} X'^\cdot \longrightarrow Y^\cdot[1],$$
$$X^\cdot \xrightarrow{gf} Z^\cdot \xrightarrow{l} Y'^\cdot \longrightarrow X^\cdot[1],$$

there exists a d.t.

$$Z'^\cdot \xrightarrow{u} Y'^\cdot \xrightarrow{v} X'^\cdot \xrightarrow{w} Z'^\cdot$$

such that the following diagram commutes.

$$\begin{tikzcd}
X^\cdot \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & Y^\cdot \arrow[r, "h"] \arrow[d, "g"'] & Z'^\cdot \arrow[r] \arrow[d, dashed, "u"'] & X^\cdot[1] \arrow[d, "\mathrm{Id}"'] \\
X^\cdot \arrow[r, "gf"] \arrow[d, "f"'] & Z^\cdot \arrow[r, "l"] \arrow[d, "\mathrm{Id}"'] & Y'^\cdot \arrow[r] \arrow[d, dashed, "v"'] & X^\cdot[1] \arrow[d, "f[1]"'] \\
Y^\cdot \arrow[r, "g"] \arrow[d, "h"'] & Z^\cdot \arrow[r, "k"] \arrow[d, "l"'] & X'^\cdot \arrow[r] \arrow[d, "\mathrm{Id}"'] & Y^\cdot[1] \arrow[d, "h[1]"'] \\
Z'^\cdot \arrow[r, dashed, "u"] & Y'^\cdot \arrow[r, dashed, "v"] & X'^\cdot \arrow[r, dashed, "w"] & Z'^\cdot[1]
\end{tikzcd}$$

Proof
See [KS06, THEOREM 11.2.6].

Triangulated Categories

We define a *triangulated category* to be an additive category with translation (K, T) together with a collection of triangles, called *distinguished*, satisfying axioms TR0 – TR5 above. The above theorem states precisely that $\mathsf{K}(\mathcal{A})$ is a triangulated category. A *functor of triangulated categories* (or more simply a Δ-functor) is an additive functor of categories with translation sending distinguished triangles into distinguished triangles. A *triangulated subcategory* of a triangulated category (K, T) is a subcategory K' which is triangulated with translation T' such that T' is the restriction of T and the inclusion is a Δ-functor. We have that $\mathbf{K}^b(\mathcal{A})$ is a triangulated subcategory of both $\mathbf{K}^+(\mathcal{A})$ and $\mathbf{K}^-(\mathcal{A})$, which in turn are triangulated subcategories of $\mathbf{K}(\mathcal{A})$. We notice that if K is triangulated then K° is also triangulated.

Given a triangle in a triangulated category K

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{[1]},$$

by applying TR1 and TR4 we obtain a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & TX \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & TX \end{tikzcd}$$

from which it follows that $gf = 0$.

A *cohomological functor* is an additive functor $H : \mathsf{K} \to \mathcal{A}$ between a triangulated category and an abelian category such that for any d.t. *Cohomological Functors*

$$X \to Y \to Z \to TX$$

the sequence

$$FX \to FY \to FZ$$

is exact. Note that, applying TR3, if $X \to Y \to Z \to TX$ is distinguished and H is cohomological, we have a long exact sequence

$$\cdots \to FT^{-1}Z \to FX \to FY \to FY \to FZ \to FTX \to \cdots$$

I.2.12 Proposition

Given an object $W \in \mathsf{K}$ *of a triangulated category, the functors* $\mathrm{Hom}_{\mathsf{K}}(W, -)$ *and* $\mathrm{Hom}_{\mathsf{K}}(-, W)$ *are cohomological.*

Proof

Let

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow TX$$

be a distinguished triangle. We want to show that the sequence

$$\mathrm{Hom}(W, X) \longrightarrow \mathrm{Hom}(W, Y) \longrightarrow \mathrm{Hom}(W, Z)$$

is exact. It is obviously a complex since $gf = 0$. Let now $\phi \in \mathrm{Hom}(W, Y)$ be a morphism such that $g\phi = 0$. We must show that there exists $\psi \in \mathrm{Hom}(W, X)$ such that $f\psi = \phi$. In other words we want to show the existence of the dashed arrows in the following diagram

$$\begin{tikzcd}
W \arrow[r, "\mathrm{Id}"] \arrow[d, dashed, "\psi"'] & W \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & TW \arrow[d, dashed, "T\psi"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & TX
\end{tikzcd}$$

which is assured by TR3 and TR4. For $\mathrm{Hom}(-, W)$ one proceeds analogously in K°. ∾

I.2.13 Proposition
Consider a morphism of distinguished triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & TX \arrow[d, "T\alpha"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & TX'
\end{tikzcd}$$

if α *and* β *are isomorphisms then so is* γ.

Proof
We apply $\mathrm{Hom}(W, -)$ to the diagram above and we write $\widetilde{A}$ for $\mathrm{Hom}(W, A)$ and $\widetilde{l}$ for $\mathrm{Hom}(W, l)$.

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{\alpha}"'] & \widetilde{Y} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{\beta}"'] & \widetilde{Z} \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{\gamma}"'] & \widetilde{TX} \arrow[r, "\widetilde{Tf}"] \arrow[d, "\widetilde{T\alpha}"'] & \widetilde{TY} \arrow[d, "\widetilde{T\beta}"'] \\
\widetilde{X'} \arrow[r, "\widetilde{f'}"] & \widetilde{Y'} \arrow[r, "\widetilde{g'}"] & \widetilde{Z'} \arrow[r, "\widetilde{h'}"] & \widetilde{TX'} \arrow[r, "\widetilde{Tf'}"] & \widetilde{TY'}
\end{tikzcd}$$

Since $\mathrm{Hom}(W, -)$ is cohomological the top and bottom rows are exact, also all vertical arrows, except perhaps for $\widetilde{\gamma}$, are isomorphisms. As a consequence of the five lemma we have that $\widetilde{\gamma}$ also is an isomorphism.

The arrow $Z \to Z'$ gives rise to a morphism of functors

$$\mathrm{Hom}(-, Z) \xrightarrow{h_\gamma} \mathrm{Hom}(-, Z').$$

From the discussion above we know that for any W the map $\mathrm{Hom}(W, Z) \to \mathrm{Hom}(W, Z')$ is an isomorphism, therefore h_γ is an isomorphism of functors. Finally, applying the Yoneda lemma, we have that γ is an isomorphism as well. ∾

Later we will need the following simple lemma.

I.2.14 Lemma
Let

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

be a distinguished triangle. Then f *is an isomorphism if and only if* $Z = 0$.

Proof
Consider the following morphism of distinguished triangles.

$$\begin{tikzcd} X \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & TX \arrow[d] \\ X \arrow[r, "f"] & Y \arrow[r] & Z \arrow[r] & TX \end{tikzcd}$$

By PROPOSITION I.2.13, if f is an isomorphism then $Z = 0$; on the other hand if $Z = 0$, then $0 \to Z$ is an isomorphism and therefore f is an isomorphism.

Complexes in Abelian Categories

Let's turn our attention again to complexes. Fix an abelian category $\mathcal{A}$. Consider a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $\mathsf{C}(\mathcal{A})$. It is straightforward to check that the map $\alpha(f) : Y^{\cdot} \to \mathrm{Mc}(f)$ is a monomorphism, the map $\beta(f) : \mathrm{Mc}(f) \to X^{\cdot}[1]$ is an epimorphism and the kernel of $\beta(f)$ is isomorphic to the image $\alpha(f)$. Thus we obtain a short exact sequence

$$0 \to Y^{\cdot} \to \mathrm{Mc}(f) \to X^{\cdot}[1] \to 0.$$

Hence, if $X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to X^{\cdot}[1]$ is a distinguished triangle, which means that it is isomorphic to a mapping cone triangle as above, we obtain a short exact sequence

$$0 \to Y^{\cdot} \to Z^{\cdot} \to X^{\cdot}[1] \to 0.$$

Using TR3 it follows that the functor $H^0 : \mathsf{K}(\mathcal{A}) \to \mathcal{A}$ is cohomological.

I.3 LOCALIZATIONS

Let C be a category and let $\mathcal{S}$ be a class of morphisms. A *localization* of C by $\mathcal{S}$ is a category $\mathsf{C}_{\mathcal{S}}$ together with a functor $Q : \mathsf{C} \to \mathsf{C}_{\mathcal{S}}$ satisfying the following axioms.

Loc1 For all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism in $\mathsf{C}_{\mathcal{S}}$.

Loc2 Any functor $F : \mathsf{C} \to \mathsf{D}$, such that $F(s)$ is an isomorphism in D, factors uniquely through $\mathsf{C}_{\mathcal{S}}$. In other words there exists a unique functor $F_{\mathcal{S}} : \mathsf{C}_{\mathcal{S}} \to \mathsf{D}$ such that F is the composition $F_{\mathcal{S}} \circ Q$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathsf{D} \\ \mathsf{C}_{\mathcal{S}} \arrow[ur, dashed, "F_{\mathcal{S}}"'] & \end{tikzcd}$$

We remark that axiom Loc2 implies that the natural map

$$\mathrm{Hom}_{\mathrm{Fct}(\mathsf{C}_{\mathcal{S}},\mathsf{D})}(G_1, G_2) \to \mathrm{Hom}_{\mathrm{Fct}(\mathsf{C},\mathsf{D})}(G_1 \circ Q, G_2 \circ Q)$$

is bijective, viz. the functor $\circ Q$ is fully faithful.

I.3.1 Remark It is true that localizations always exist (see [GM03] or [Mil]) and their construction is not hard: one simply adds formal inverses to the arrows $s \in \mathcal{S}$. Of course there are some technical details to verify, however we do not need such a general result.

The localization $\mathsf{C}_{\mathcal{S}}$ is unique up to equivalence and as a consequence we have that $(\mathsf{C}^\circ)_{\mathcal{S}^\circ}$ is equivalent to $(\mathsf{C}_{\mathcal{S}})^\circ$.

Multiplicative Systems

In the general context the localization of a category is a very complicated object: morphisms are not very easy to handle. However one gains some structure by imposing the following conditions on $\mathcal{S}$. A *left multiplicative system* in a category C is a collection of morphisms $\mathcal{S}$ satisfying the following axioms.

S1 If f is an isomorphism then f belongs to $\mathcal{S}$.

S2 If $f, g \in \mathcal{S}$ then $gf \in \mathcal{S}$, whenever defined.

S3 Given a diagram

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $s \in \mathcal{S}$, it can be completed to

$$\begin{tikzcd} W \arrow[r, dashed, "g"] \arrow[d, dashed, "t"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $t \in \mathcal{S}$.

S4 Given a commutative diagram

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "s"] & Z \end{tikzcd}$$

with $s \in \mathcal{S}$, it can be completed to

$$\begin{tikzcd} W \arrow[r, dashed, "t"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "s"] & Z \end{tikzcd}$$

where $t \in \mathcal{S}$.

Analogously a *right multiplicative system* in a category C is a class of morphisms $\mathcal{S}$ satisfying S1, S2 and the following *mirror* axioms.

S3′ Given a diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \\ W & \end{tikzcd}$$

where $s \in \mathcal{S}$, it can be completed to

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[d, dashed, "t"] \\ W \arrow[r, dashed, "g"'] & Z \end{tikzcd}$$

where $t \in \mathcal{S}$.

S4′ Given a commutative diagram

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd}$$

with $s \in \mathcal{S}$, it can be completed to

$$\begin{tikzcd} W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dashed, "t"] & Z \end{tikzcd}$$

where $t \in \mathcal{S}$.

Multiplicative systems are sometimes referred to as *localizing classes*. We are mainly interested in the case when $\mathcal{S} = \mathsf{Qis}$ is the class of quasi-isomorphisms of a triangulated subcategory of $\mathbf{K}(\mathcal{A})$, for some additive category $\mathcal{A}$. Later we show that Qis is in fact both a left and right multiplicative system.

I.3.2 Theorem

Let $\mathcal{S}$ be a left multiplicative system in a category C. Then the localization of C by $\mathcal{S}$ exists and it may be defined as follows.

$$\operatorname{Ob} \mathsf{C}_{\mathcal{S}} = \operatorname{Ob} \mathsf{C}$$
$$\operatorname{Hom}_{\mathsf{C}_{\mathcal{S}}}(X, Y) = \{(s, X', f) \mid s : X' \to X,\ f : X' \to Y,\ s \in \mathcal{S}\}\big/_{\sim}$$

where the equivalence relation $\sim$ is defined below. A triple (s, X', f) can be visualized as a left roof

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

and given two roofs (s, X', f) (t, X'', g)

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & X'' \arrow[dlll, "t"' near end] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

are equivalent if there exists a third roof (u, X''', h) *such that the following diagram commutes.*

$$\begin{tikzcd}
& & X''' \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
& X' \arrow[dl, "s"'] \arrow[drrr, "f" near end] & & X'' \arrow[dlll, "t"' near end] \arrow[dr, "g"] & \\
X & & & & Y
\end{tikzcd}$$

Given two roofs (s, X', f), (t, Y', g) *we can find, thanks to* S3, *a third roof* (u, W, h) *such that the following commutes*

$$\begin{tikzcd}
& & W \arrow[dl, dashed, "u"'] \arrow[dr, dashed, "h"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

and we define the composition of (t, Y', g) *with* (s, X', f) *by the equivalence class of the roof* (su, W, gh).

Similarly, if $\mathcal{S}$ *is a right multiplicative system, the localization* $\mathsf{C}_{\mathcal{S}}$ *exists and it may defined analogously. In particular, morphisms may be represented by* right roofs (f, Y', s)

$$\begin{tikzcd}
& Y' & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"']
\end{tikzcd}$$

under the corresponding equivalence relation and with the corresponding composition.

Sketch of Proof

The proof is rather technical, we refer to [Mil, CHAPTER 1] for all the necessary details. First one verifies that ~ is indeed an equivalence relation over roofs and that composition is well-defined, associative and that identities exist. The quotient functor is given by the identity on objects and sends any morphism $f : X \to Y$ to the roof (Id, X, f), or to the roof (f, Y, Id). If $F : \mathsf{C} \to \mathsf{D}$ is a functor sending arrows in $\mathcal{S}$ to isomorphisms, then we define $F_{\mathcal{S}} : \mathsf{C}_{\mathcal{S}} \to \mathsf{D}$ to be F on objects and to send a roof (s, X', f) to $F(f) \circ F(s)^{-1}$, or equivalently to send a roof (f, Y', s) to $F(s)^{-1} \circ F(f)$. If $G : \mathsf{C}_{\mathcal{S}} \to \mathsf{D}$ is another factorization of F then, for any roof (s, X', f),

$$\begin{aligned} G((s, X', f)) &= G((\mathrm{Id}, X', f) \circ (s, X', \mathrm{Id})) \\ &= G(Q(f)) \circ G\left(Q(s)^{-1}\right) \\ &= F(f) \circ F(s)^{-1} \\ &= F_{\mathcal{S}}((s, X', f)) . \end{aligned}$$

And similarly for right roofs. ☙

I.3.3 Convention From now on, by multiplicative system or localizing class we mean a system which is both left and right multiplicative.

Localization of Subcategories

Let's consider now the localization of a subcategory.

I.3.4 Proposition
Let C be a category, C' a full subcategory, $\mathcal{S}$ a multiplicative system. Let $\mathcal{S}'$ be the collection of morphisms of C' which belong to $\mathcal{S}$. Assume that $\mathcal{S}'$ is a multiplicative system in C', then the inclusion $\mathsf{C}' \hookrightarrow \mathsf{C}$ passes onto the localizations $\mathsf{C}'_{\mathcal{S}'} \to \mathsf{C}_{\mathcal{S}}$. Assume moreover that one of the following conditions hold

(I.4) If $s : X \to Y'$ is a morphism in $\mathcal{S}$, with $Y' \in \mathsf{C}'$, there exists $t : W' \to X$, with $W' \in \mathsf{C}'$ and such that $st \in \mathcal{S}$.

(I.5) If $s : X' \to Y$ is a morphism in $\mathcal{S}$, with $X' \in \mathsf{C}'$, there exists $t : Y \to Z'$, with $Z' \in \mathsf{C}'$ and such that $ts \in \mathcal{S}$.

Then the induced functor $\mathsf{C}'_{\mathcal{S}'} \hookrightarrow \mathsf{C}_{\mathcal{S}}$ is fully faithful.

Proof
The first statement is obvious. If $\iota : \mathsf{C}' \hookrightarrow \mathsf{C}$ and $Q : \mathsf{C} \to \mathsf{C}_{\mathcal{S}}$ are respectively the inclusion and quotient functors, then any $s \in \mathcal{S}'$ maps to $Q \circ \iota(s)$, which as an isomorphism in $\mathsf{C}_{\mathcal{S}}$, therefore $Q\iota$ factors through $\mathsf{C}'_{\mathcal{S}'}$. For the second assertion we refer to [Mil, Propositon 1.4.1]. ☙

Localization of Triangulated Categories

Let's add some structure: let K be a triangulated category. A *null system* N in K is a class of objects satisfying the following axioms.

NS1 The zero object of K belongs to N.

NS2 An object X belongs to N if and only if TX belongs to N.

NS3 If $X \to Y \to Z \to TX$ is a d.t. and $X, Y \in \mathsf{N}$, then $Z \in \mathsf{N}$.

We define $\mathcal{S}_{\mathsf{N}}$ to be the collection of morphisms $f : X \to Y$ such that f is embedded in a d.t.

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

with $Z \in \mathsf{N}$. Not only do we claim that $\mathcal{S}_{\mathsf{N}}$ is a multiplicative system, we also claim it is *compatible with the triangulation*, in the sense that it satisfies the additional axioms:

S5 For any morphism s, $s \in \mathcal{S}_{\mathcal{N}}$ if and only if $Ts \in \mathcal{S}_{\mathcal{N}}$;

S6 The following diagram, where the rows are d.t.s and the vertical maps are in $\mathcal{S}_{\mathsf{N}}$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"'] & Z \arrow[r] & TX \arrow[d, "Ts"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

can be completed into a morphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "s"'] & Y \arrow[r] \arrow[d, "t"'] & Z \arrow[r] \arrow[d, dashed, "u"'] & TX \arrow[d, "Ts"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

with $u \in \mathcal{S}_{\mathsf{N}}$.

Proof
See [KS90, Proposition 1.6.7]. ❧

In this case we denote by K/N the localization $\mathsf{K}_{\mathcal{S}_{\mathsf{N}}}$. The key result is that K/N is a triangulated category and that the quotient functor $Q : \mathsf{K} \to \mathsf{K}/\mathsf{N}$ is a Δ-functor. In fact, if we start with a localizing class $\mathcal{S}$ compatible with the triangulation, we observe that there is an obvious translation functor on $\mathsf{K}_{\mathcal{S}}$. Indeed, if $s \in \mathcal{S}$ then by S5 $Ts \in \mathcal{S}$, therefore T induces an automorphism $T_{\mathcal{S}} : \mathsf{K}_{\mathcal{S}} \to \mathsf{K}_{\mathcal{S}}$ which we denote simply by T. Since $\mathsf{K}_{\mathcal{S}}$, endowed with the translation T, is a category with translation, we may speak of triangles in $\mathsf{K}_{\mathcal{S}}$. We define a triangle $X \to Y \to Z \to TX$ of $\mathsf{K}_{\mathcal{S}}$ to be distinguished if there exists a d.t. $X' \to Y' \to Z' \to TX'$, of K, and an isomorphism of triangles

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"'] & Y \arrow[r] \arrow[d, "\beta"'] & Z \arrow[r] \arrow[d, "\gamma"'] & TX \arrow[d, "T\alpha"'] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & TX' \end{tikzcd}$$

in $\mathsf{K}_{\mathcal{S}}$. With the above translation and the above collection of distinguished triangles the category $\mathsf{K}_{\mathcal{S}}$ becomes triangulated and the quotient functor $Q : \mathsf{K} \to \mathsf{K}_{\mathcal{S}}$ is a Δ-functor (see [Mil, Theorem 1.6.1]).

Again, if N is a null system and $X \in \mathsf{N}$ we have $Q(X) \simeq 0$. In fact $X \to X \to 0 \to TX$ is a d.t. and therefore $X \to 0 \to TX \to X$ is a d.t. and $TX \in \mathsf{N}$, thus $(X \to 0) \in \mathcal{S}_{\mathsf{N}}$ yielding $Q(X) \simeq 0$.

Furthermore K/N satisfies the following universal property: any Δ-functor functor $F : \mathsf{K} \to \mathsf{K}'$ such that $FX \simeq 0$ for all $X \in \mathsf{N}$, factors uniquely through K/N. In fact if $f \in \mathcal{S}_{\mathsf{N}}$, namely

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$$

is a d.t. with $Z \in \mathsf{N}$, then

$$\mathrm{FX} \xrightarrow{\mathrm{F}f} \mathrm{FY} \longrightarrow \mathrm{FZ} \longrightarrow \mathrm{TFX}$$

is a d.t. We then consider the morphism of d.t.s

$$\begin{tikzcd}
\mathrm{FX} \arrow[r] \arrow[d, "\mathrm{F}f"'] & \mathrm{FY} \arrow[r] \arrow[d, "\mathrm{Id}"'] & \mathrm{FZ} \arrow[r] \arrow[d] & \mathrm{TFX} \arrow[d, "\mathrm{TF}f"'] \\
\mathrm{FY} \arrow[r] & \mathrm{FY} \arrow[r] & 0 \arrow[r] & \mathrm{TFY}
\end{tikzcd}$$

and since Id and $(\mathrm{FZ} \to 0)$ are both isomorphisms so is $\mathrm{F}f$. As a result F factors through K/N.

I.3.5 Theorem

Let K be a triangulated category; N a null system in K; K' a full triangulated subcategory; and finally let N' be $\mathsf{N} \cap \mathsf{K}'$. Assume that

> *(I.6) any d.t. in K, $X' \to Y' \to Z \to TX$, with $X, Y \in \mathsf{K}'$, is isomorphic to a d.t. in K',*

then N' is a null system in K' and the inclusion $\mathsf{K}' \hookrightarrow \mathsf{K}$ induces a functor $\mathsf{K}'/\mathsf{N}' \to \mathsf{K}/\mathsf{N}$. Assume moreover the following condition.

> *(I.7) Any morphism $X' \to N$ in K, with $Y \in \mathsf{K}'$ and $Z \in \mathsf{N}$, factors through an object of N'.*

Then the induced functor $\mathsf{K}'/\mathsf{N}' \to \mathsf{K}/\mathsf{N}$ is fully faithful.

Proof

See [KS90, Proposition 1.6.10].

I.4 DERIVED CATEGORIES

Fix an abelian category $\mathcal{A}$. Let $\mathbf{N}^*(\mathcal{A})$ be the family of complexes in $\mathbf{K}^*(\mathcal{A})$ quasi-isomorphic to zero, where $* = \mathsf{ub}, +, -, \mathsf{b}$. It is straightforward to verify that $\mathbf{N}^*(\mathcal{A})$ is a null system, in fact: axioms NS1 and NS2 are obviously satisfied, and NS3 is a consequence of the fact that cohomology is a cohomological functor. We now show that the localizing class $\mathcal{S}$ associated with $\mathbf{N}^*(\mathcal{A})$ is precisely $\mathsf{Qis}^*(\mathcal{A})$, the class of quasi-isomorphisms in the category $\mathbf{K}^*(\mathcal{A})$. If $f \in \mathcal{S}$ then there is a d.t.

$$X^\cdot \xrightarrow{f} Y^\cdot \longrightarrow Z^\cdot \longrightarrow X^\cdot[1]$$

where $Z^\cdot$ is quasi-isomorphic to zero. Since $H^\cdot$ is cohomological we have an exact sequence $H^{n-1}(Z^\cdot) \to H^n(X^\cdot) \to H^n(Y^\cdot) \to H^n(Z^\cdot)$ from which it follows that $H^n(X^\cdot) \to H^n(Y^\cdot)$ is an isomorphism, and hence f is a qis. On the other hand, if f is a qis then the cohomology long exact sequence associated with the d.t.

$$X^\cdot \xrightarrow{f} Y^\cdot \longrightarrow \mathrm{Mc}(f) \longrightarrow X^\cdot[1]$$

implies that the mapping cone $\mathrm{Mc}(f)$ is quasi-isomorphic to zero, thus proving $f \in \mathcal{S}$.

The derived Category

We define the *derived category* of $\mathcal{A}$ to be $\mathbf{D}(\mathcal{A}) = \mathbf{K}(\mathcal{A})/\mathbf{N}(\mathcal{A})$, along with the bounded versions $\mathbf{D}^*(\mathcal{A}) = \mathbf{K}^*(\mathcal{A})/\mathbf{N}^*(\mathcal{A})$, for $* = +, -, \mathrm{b}$. If the category $\mathcal{A}$ is understood we often omit to explicit the dependency from $\mathcal{A}$ in the notation. The results of the previous section imply that $\mathbf{D}$ is triangulated and that the quotient functor $Q : \mathbf{K} \to \mathbf{D}$ is a Δ-functor.

I.4.1 Remark If $Q : \mathbf{K} \to \mathbf{D}$ is the quotient functor and $X^{\cdot} \in \mathbf{K}$ then:

$$Q(X) \simeq 0 \iff H^{\cdot}(X^{\cdot}) = 0.$$

Also a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $\mathbf{D}$ is an isomorphism if and only if there exists a qis $s : W^{\cdot} \to X^{\cdot}$ such that fs is null homotopic; or dually, f is an isomorphism in $\mathbf{D}$ if and only if there exists a qis $t : Y^{\cdot} \to Z^{\cdot}$ such that tf is null homotopic.

The cohomology functors pass over to the derived category. Namely the functor $H^n : \mathbf{D}(\mathcal{A}) \to \mathcal{A}$ is well-defined and cohomological. Of course, the functor $H^{\cdot} : \mathbf{D} \to \mathbf{C}$ is also well-defined. We recall that, given a complex $X^{\cdot}$, $H^{\cdot}(X^{\cdot})$ is defined to be the complex $(H^n(X^{\cdot}))_n$ with null differentials. Also, a morphism $f : X^{\cdot} \to Y^{\cdot}$ in $\mathbf{D}$ is an isomorphism if and only if $H^n(f)$ is an isomorphism for all $n \in \mathbb{Z}$.

If we start with a short exact sequence

$$0 \to X^{\cdot} \to Y^{\cdot} \to Z^{\cdot} \to 0$$

we define a morphism of complexes $\varphi : \mathrm{Mc}(f) \to Z^{\cdot}$ as $\varphi = (0, g)$. We claim that φ is a quasi-isomorphism. There is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X^{\cdot} \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & X^{\cdot} \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X^{\cdot} \arrow[r, "f"] & Y^{\cdot} \arrow[r] & Z^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

which yields a short exact sequence

$$0 \to \mathrm{Mc}(\mathrm{Id}_{X^{\cdot}}) \to \mathrm{Mc}(f) \to \mathrm{Mc}(0 \to Z^{\cdot}) \to 0$$

where $\mathrm{Mc}(0 \to Z^{\cdot})$ is clearly equal to $Z^{\cdot}$ and it is easy to see that $\mathrm{Mc}(\mathrm{Id}_{X^{\cdot}})$ is an exact complex. As a consequence of the cohomology long exact sequence we have isomorphisms $H^n(\mathrm{Mc}(f)) \xrightarrow{\sim} H^n(Z^{\cdot})$. If we denote the inverse of φ in $\mathbf{D}$ by ψ we have a morphism of triangles

$$\begin{tikzcd}
X^{\cdot} \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & Y^{\cdot} \arrow[r, "\alpha(f)"] \arrow[d, "\mathrm{Id}"'] & \mathrm{Mc}(f) \arrow[r, "\beta(f)"] \arrow[d, "\varphi"'] & TX^{\cdot} \arrow[d, "\mathrm{Id}"'] \\
X^{\cdot} \arrow[r, "f"] & Y^{\cdot} \arrow[r, "g"] & Z^{\cdot} \arrow[r, "\psi \circ \beta(f)"] & TX^{\cdot}
\end{tikzcd}$$

hence an exact sequence gives rise to a distinguished triangle in $\mathbf{D}$.

Bounded Cohomology

We now illustrate another property of derived categories. Let's denote by $\mathbf{D}_*$ the full triangulated subcategory of $\mathbf{D}$ consisting of complexes $X^\cdot$ such that $H^\cdot(X^\cdot)$ belongs to $\mathbf{C}^*$ (for $* = +, -, b$). With a slight abuse of notation, we claim that $\mathbf{D}^*$, the full subcategory consisting of complexes lying in to $\mathbf{C}^*$, is equivalent via the inclusion to $\mathbf{D}_*$, the full subcategory consisting of complexes with *cohomology* lying in $\mathbf{C}^*$(see [KS06]).

We say that a complex $X^\cdot$ is a *k-complex* if $X^n = 0$ for $n \neq k$. We say that $X^\cdot$ is an *H^k-complex* if $H^n(X^\cdot) = 0$ for $n \neq k$. We implicitly think of objects of $\mathcal{A}$ as 0-complexes. One has that, through the composition $\mathcal{A} \to \mathbf{C} \to \mathbf{K} \to \mathbf{D}$, $\mathcal{A}$ is equivalent the full subcategory of $\mathbf{D}$ consisting of H^0-complexes.

I.5 RESOLUTIONS

We recall that if $\mathcal{B}$ is a subcategory of $\mathcal{A}$ we say that $\mathcal{B}$ is *cogenerating* if any $A \in \mathcal{A}$ is a subobject of some $B \in \mathcal{B}$. We say that $\mathcal{B}$ is *generating* if any $A \in \mathcal{A}$ is a quotient of some $B \in \mathcal{B}$.

I.5.1 Lemma
Let $\mathcal{I}$ be a full cogenerating additive subcategory of $\mathcal{A}$ and let $X^\cdot \in \mathbf{C}^{\geq a}(\mathcal{A})$, for some integer $a \in \mathbb{Z}$. Then there exists $I^\cdot \in \mathbf{C}^{\geq a}(\mathcal{I})$ and a qis $X^\cdot \to I^\cdot$.

Dually, if $\mathcal{P}$ is a full generating additive subcategory of $\mathcal{A}$ and $X^\cdot \in \mathbf{C}^{\leq a}(\mathcal{A})$, then there exists $P^\cdot \in \mathbf{C}^{\leq a}(\mathcal{P})$ and a qis $P^\cdot \to X^\cdot$.

Proof
See [KS06, Proposition 13.2.1].

Again, let $\mathcal{I}$ and $\mathcal{P}$ be two additive subcategories of $\mathcal{A}$. Consider the following conditions.

$$\left\{\begin{array}{l}\text{There exists an integer } d \geq 0 \text{ such} \\ \text{that, for any exact sequence} \\ \qquad I_d \to \cdots \to I_1 \to X \to 0 \\ \text{with } I_j \in \mathcal{J},\ Y \in \mathcal{J}.\end{array}\right. \tag{I.8}$$

$$\left\{\begin{array}{l}\text{There exists an integer } d \geq 0 \text{ such} \\ \text{that, for any exact sequence} \\ \qquad 0 \to X \to P_1 \to \cdots \to P_d \\ \text{with } P_j \in \mathcal{P},\ X \in \mathcal{P}.\end{array}\right. \tag{I.9}$$

I.5.2 Theorem
If $\mathcal{I}$ is a full cogenerating subcategory of $\mathcal{A}$ then the natural functor

$$\mathbf{K}^+(\mathcal{I})/\mathbf{N}^+(\mathcal{I}) \longrightarrow \mathbf{D}^+(\mathcal{A})$$

is an equivalence. Moreover, if $\mathcal{I}$ satisfies condition (I.8) above, then the functor

$$\mathbf{K}^b(\mathcal{I})/\mathbf{N}^b(\mathcal{I}) \longrightarrow \mathbf{D}^b(\mathcal{A})$$

is also an equivalence.

Similarly, if $\mathcal{P}$ is a full generating subcategory of $\mathcal{A}$ then the functor

$$\mathbf{K}^{-}(\mathcal{P})/\mathbf{N}^{-}(\mathcal{P}) \longrightarrow \mathbf{D}^{-}(\mathcal{A})$$

is an equivalence. Moreover, if $\mathcal{P}$ satisfies condition (I.9) above, then the functor

$$\mathbf{K}^{b}(\mathcal{P})/\mathbf{N}^{b}(\mathcal{P}) \longrightarrow \mathbf{D}^{b}(\mathcal{A})$$

is also an equivalence.

Injectives and Projectives

We now introduce injective and projective objects of an abelian category $\mathcal{A}$. We recall that, given $X \in \mathcal{A}$, the functors $\mathrm{Hom}_{\mathcal{A}}(-, X)$ and $\mathrm{Hom}_{\mathcal{A}}(X, -)$ are both left exact. We say that an object $I \in \mathcal{A}$ is *injective* (in $\mathcal{A}$) if the functor $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact. Analogously, an object P is *projective* if $\mathrm{Hom}_{\mathcal{A}}(P, -)$ is exact. We say that $\mathcal{A}$ *has enough injectives* if the full subcategory $\mathcal{I}_{\mathcal{A}}$ consisting of all injective objects is cogenerating. Of course, we say that $\mathcal{A}$ *has enough projectives* if the full subcategory $\mathcal{P}_{\mathcal{A}}$ consisting of all projective objects is generating.

There other equivalent ways to define injective and projective objects. First we give another definition.

I.5.3 DEFINITION/PROPOSITION

Let

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0$$

be a short exact sequence. Then the following are equivalent.

(I.10) There exists $h : X'' \to X$ such that $gh = \mathrm{Id}_{X''}$.

(I.11) There exists $k : X \to X'$ such that $kf = \mathrm{Id}_{X'}$.

(I.12) There exist $h : X'' \to X$ and $k : X \to X'$ such that $\mathrm{Id}_X = fk + hg$.

(I.13) There exist $\varphi : X \to X' \oplus X''$ and $\psi : X' \oplus X'' \to X$ such that φ and ψ are mutually inverses of each other.

(I.14) For any $Y \in \mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(Y, g)$ is surjective.

(I.15) For any $Y \in \mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(f, Y)$ is surjective.

If any of the above conditions holds, we say that the short exact sequence splits.

Proof

See [KS06, PROPOSITION 8.3.14]. ☙

I.5.4 THEOREM

The following are equivalent.

(I.16) I is injective.

(I.17) Maps $X' \to I$, where X' is a subobject of some object X, can be extended to X. In other words

$$\begin{tikzcd} 0 \arrow[r] & X' \arrow[r] \arrow[dr] & X \arrow[d, dashed] \\ & & I \end{tikzcd}$$

we claim the existence of the dashed arrow in the above diagram with exact rows.

(I.18) Any exact sequence

$$0 \to I \to X \to X'' \to 0$$

splits.

Also, any two objects X', X'' *are injective if and only if* $X' \oplus X''$ *is injective.*

Similarly, the following are equivalent.

(I.19) P *is injective.*

(I.20) Maps $P \to X''$*, where* X'' *is a quotient of some object* X*, can be lifted to* X*. In other words*

$$\begin{tikzcd} P \arrow[d, dashed] \arrow[dr] & & \\ X \arrow[r] & X'' \arrow[r] & 0 \end{tikzcd}$$

we claim the existence of the dashed arrow in the above diagram with exact rows.

(I.21) Any exact sequence

$$0 \to X' \to X \to P \to 0$$

splits.

Also, any two objects X', X'' *are projective if and only if* $X' \oplus X''$ *is projective.*

Proof

See [KS06, Section 8.4].

We state an important technical property of injectives and projectives.

I.5.5 Lemma

If $f : X^{\cdot} \to I^{\cdot}$ *is a morphism in* $\mathbf{C}(\mathcal{A})$*, where* $I^{\cdot} \in \mathcal{I}_{\mathcal{A}}$ *is a complex made up of injectives and where* $X^{\cdot}$ *is an exact complex, then* f *is null homotopic.*

Dually, if $g : P^{\cdot} \to X^{\cdot}$ *is a morphism in* $\mathbf{C}(\mathcal{A})$*, where* $P^{\cdot} \in \mathcal{P}_{\mathcal{A}}$ *is a complex made up of projectives and where* $X^{\cdot}$ *is an exact complex, then* g *is null homotopic.*

Proof

See [KS06, Lemma 13.2.4].

A great consequence of the above lemma is the following theorem. We remind that we denote $\mathcal{I}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}}$ the full subcategories of $\mathcal{A}$, consisting of injectives and projectives respectively.

I.5.6 Theorem
If $\mathcal{A}$ has enough injectives then the quotient functor $\mathsf{K}^+(\mathcal{I}_{\mathcal{A}}) \to \mathsf{D}^+(\mathcal{A})$ is an equivalence. If moreover $\mathcal{I}_{\mathcal{A}}$ satisfies condition (I.8) above then the quotient functor $\mathsf{K}^{\mathrm{b}}(\mathcal{I}_{\mathcal{A}}) \to \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is also an equivalence.

Dually, if $\mathcal{A}$ has enough projectives then the quotient functor $\mathsf{K}^-(\mathcal{P}_{\mathcal{A}}) \to \mathsf{D}^-(\mathcal{A})$ is an equivalence. If moreover $\mathcal{P}_{\mathcal{A}}$ satisfies condition (I.9) above then the quotient functor $\mathsf{K}^{\mathrm{b}}(\mathcal{P}_{\mathcal{A}}) \to \mathsf{D}^{\mathrm{b}}(\mathcal{A})$ is also an equivalence.

Derived Category of Subcategories

We now consider the localization of an abelian subcategory of $\mathcal{A}$.

I.5.7 Notation If $\mathcal{A}'$ is a full abelian subcategory of $\mathcal{A}$ we write $\mathsf{D}^*_{\mathcal{A}'}(\mathcal{A})$ to indicate the full triangulated subcategory of $\mathsf{D}^*(\mathcal{A})$ consisting of complexes whose cohomology lies in $\mathcal{A}'$, where $* = \mathrm{ub}, +, -, \mathrm{b}$.

Of course the inclusion $\mathcal{A}' \to \mathcal{A}$ induces a functor $\mathsf{D}^*(\mathcal{A}') \to \mathsf{D}^*_{\mathcal{A}'}(\mathcal{A})$. We recall that a full abelian subcategory $\mathcal{B}$ is *thick* if it is closed under kernels, cokernels and extensions.

I.5.8 Theorem
Let $\mathcal{A}'$ be a thick subcategory of $\mathcal{A}$. Assume the following condition.

(I.22) For any monomorphism $\mathrm{W}' \hookrightarrow \mathrm{X}$, with $\mathrm{W}' \in \mathcal{A}'$, there exists a morphism $\mathrm{X} \to \mathrm{I}'$, with $\mathrm{I}' \in \mathcal{A}'$, such that the composition is also a monomorphism (it can be visualized by the following diagram with exact rows and diagonals).

$$\begin{tikzcd}
0 \arrow[dr] & & \\
0 \arrow[r] & \mathrm{W}' \arrow[r] \arrow[dr] & \mathrm{X} \arrow[d, dashed] \\
& & \mathrm{I}'
\end{tikzcd}$$

Then the functors $\mathsf{D}^+(\mathcal{A}') \to \mathsf{D}^+_{\mathcal{A}'}(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A}') \to \mathsf{D}^{\mathrm{b}}_{\mathcal{A}'}(\mathcal{A})$ are equivalences.

Similarly, we assume the following condition.

(I.23) For any epimorphism $\mathrm{X} \twoheadrightarrow \mathrm{Z}'$, with $\mathrm{Z}' \in \mathcal{A}'$, there exists a morphism $\mathrm{P}' \to \mathrm{X}$, with $\mathrm{P}' \in \mathcal{A}'$, such that the composition is also an epimorphism (it can be visualized by the following diagram with exact rows and diagonals).

$$\begin{tikzcd}
P' \arrow[d, dashed] \arrow[dr] & & \\
X \arrow[r] & Z' \arrow[r] \arrow[dr] & 0 \\
& & 0
\end{tikzcd}$$

Then the functors $\mathbf{D}^-(\mathcal{A}') \to \mathbf{D}^-_{\mathcal{A}'}(\mathcal{A})$ *and* $\mathbf{D}^b(\mathcal{A}') \to \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$ *are equivalences.*

Proof
See [KS06, THEOREM 13.2.8].

As a particular case we obtain the next corollary.

I.5.9 COROLLARY
If $\mathcal{A}'$ *is a thick subcategory of* $\mathcal{A}$ *such that any* $X' \in \mathcal{A}'$ *is a subobject of an* $I' \in \mathcal{A}'$ *which is injective as an object of* $\mathcal{A}$*, then the functors* $\mathbf{D}^+(\mathcal{A}') \to \mathbf{D}^+_{\mathcal{A}'}(\mathcal{A})$ *and* $\mathbf{D}^b(\mathcal{A}') \to \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$ *are equivalences.*

Analogously, if any $X' \in \mathcal{A}'$ *is a quotient of a* $P' \in \mathcal{A}'$ *which is projective as an object of* $\mathcal{A}$*, then the functors* $\mathbf{D}^-(\mathcal{A}') \to \mathbf{D}^-_{\mathcal{A}'}(\mathcal{A})$ *and* $\mathbf{D}^b(\mathcal{A}') \to \mathbf{D}^b_{\mathcal{A}'}(\mathcal{A})$ *are equivalences.*

I.6 DERIVED FUNCTORS

Localization of Functors

Let us start by considering the general situation. Consider a functor $F : \mathsf{C} \to \mathsf{D}$ and a class of morphisms $\mathcal{S}$ in C. We already know that if F sends elements of $\mathcal{S}$ to isomorphisms in D then the functor factors through the localization. In the general case however F need not to factor, so we want to define what resembles most closely the desired factorization. We give the following definitions.

(I.24) A *right localization* of F (with respect to $\mathcal{S}$) is a functor $\mathbf{R}_{\mathcal{S}}F : \mathsf{C}_{\mathcal{S}} \to \mathsf{D}$ together with a morphism of functors $\tau : F \to \mathbf{R}_{\mathcal{S}}F \circ Q$

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "\mathbf{R}_{\mathcal{S}}F \circ Q"'] \arrow[r, phantom, "\Downarrow \tau"] & \mathsf{D}
\end{tikzcd}$$

such that for any functor $G : \mathsf{C}_{\mathcal{S}} \to \mathsf{D}$ the map

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Fct}(\mathsf{C}_{\mathcal{S}},\mathsf{D})}(\mathbf{R}_{\mathcal{S}}F, G) &\longrightarrow \mathrm{Hom}_{\mathrm{Fct}(\mathsf{C},\mathsf{D})}(\mathbf{R}_{\mathcal{S}}F \circ Q, G \circ Q) \\
&\xrightarrow{\circ\tau} \mathrm{Hom}_{\mathrm{Fct}(\mathsf{C},\mathsf{D})}(F, G \circ Q)
\end{aligned}$$

is bijective. A functor F is said to be *right localizable* if it admits a right localization. Note that if $(\mathbf{R}_{\mathcal{S}}F, \tau)$ exists it is unique up to unique isomorphism.[3]

3 Given a second localization $(\mathbf{R}', \tau')$, substitute G with $\mathbf{R}'$ and trace back τ and τ' to obtain the isomorphism.

(I.25) $F : C \to D$ is a *universally right localizable* functor is such that for any $F' : D \to D'$, the functor $F' \circ F$ is right localizable and the obvious[4] natural transformation

$$\mathbf{R}_{\mathcal{S}}(F' \circ F) \xrightarrow{\sim} F' \circ \mathbf{R}_{\mathcal{S}} F$$

is an isomorphism.

Similarly one defines left localizations. A *left localization* of F is a functor $\mathbf{L}_{\mathcal{S}} F : C_{\mathcal{S}} \to D$ together with a morphism of functors $\sigma : \mathbf{L}_{\mathcal{S}} F \circ Q \to F$

$$\begin{tikzcd} C \arrow[r, bend left, "\mathbf{L}_{\mathcal{S}} F \circ Q"{name=U}] \arrow[r, bend right, "F"'{name=D}] & D \arrow[from=U, to=D, phantom, "\Downarrow \sigma"] \end{tikzcd}$$

such that for any other functor $G : C_{\mathcal{S}} \to D$ the natural transformation σ induces a bijection

$$\mathrm{Hom}_{\mathrm{Fct}(C_{\mathcal{S}}, D)}(G, \mathbf{L}_{\mathcal{S}} F) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fct}(C, D)}(G \circ Q, F).$$

We say that F is *left localizable* if it admits a left localization; of course, the pair $(\mathbf{L}_{\mathcal{S}} F, \sigma)$ is unique up to unique isomorphism. Finally F is *universally left localizable* if for any functor $F' : D \to D'$, $F' \circ F$ is left localizable and the natural transformation $F' \circ \mathbf{L}_{\mathcal{S}} F \to \mathbf{L}_{\mathcal{S}}(F' \circ F)$ is an isomorphism.

I.6.1 Caution Even if a given functor F is both left and right localizable the localizations $\mathbf{L}F$ and $\mathbf{R}F$ are in general not isomorphic. However there always exists a morphism $\mathbf{L}F \to \mathbf{R}F$ between them.

I.6.2 Notation In various definitions that we encounter one often has to explicit the dependency over some localizing class $\mathcal{S}$ or some null system N. As it becomes cumbersome we choose to omit the dependencies in the notations when no confusion may arise. We remind that we are mainly interested in only one type of multiplicative systems: quasi-isomorphisms.

Admittedly, the above definitions are rather technical and abstract. We would like an easier way to establish the existence and to compute the localizations of a functor. It turns out that there are some cases in which computing the localizations becomes easier.

I.6.3 Theorem
Let $F : C \to D$ be a functor. Let $\mathcal{S}$ be a multiplicative system in C, let C' be a subcategory of C and denote by $\mathcal{S}'$ the class of morphisms of C' belonging to $\mathcal{S}$. Assume that for any $s \in \mathcal{S}'$, $F(s)$ is an isomorphism in D. Consider the following conditions.

(I.26) For any $X \in C$, there exists $s : X \to I$ with $I \in C'$ and $s \in \mathcal{S}'$.

(I.27) For any $X \in C$, there exists $t : P \to X$, with $P \in C'$ and $t \in \mathcal{S}'$.

Then

4 Induced by $K\tau$.

(I.28) If (I.26) holds then F *is universally right localizable and the composition*

$$\mathsf{C}' \hookrightarrow \mathsf{C} \xrightarrow{\mathrm{Q}} \mathsf{C}_{\mathcal{S}} \xrightarrow{\mathbf{R}_{\mathcal{S}}\mathrm{F}} \mathsf{D}$$

is isomorphic to the restriction of F *to* C'.

(I.29) If (I.27) holds then F *is universally left localizable and the composition*

$$\mathsf{C}' \hookrightarrow \mathsf{C} \xrightarrow{\mathrm{Q}} \mathsf{C}_{\mathcal{S}} \xrightarrow{\mathbf{L}_{\mathcal{S}}\mathrm{F}} \mathsf{D}$$

is isomorphic to the restriction of F *to* C'.

Proof
See [KS06, PROPOSITION 7.3.2]. ❧

Localization of Triangulated Functors

We now turn to the localization of functors in triangulated categories. The definition of the localization of a Δ-functor is given exactly as above, only replacing *functor* with Δ-*functor* everywhere. Fix now two triangulated categories K, K' and two null systems N, N'. Let $\mathrm{Q} : \mathsf{K} \to \mathsf{K}/\mathsf{N}$ and $\mathrm{Q}' : \mathsf{K}' \to \mathsf{K}'/\mathsf{N}'$ be the quotient functors.

We say that a Δ-functor $\mathrm{F} : \mathsf{K} \to \mathsf{K}'$ is *right localizable* (with respect to $(\mathsf{N}, \mathsf{N}')$) if $\mathrm{Q}' \circ \mathrm{F} : \mathsf{K} \to \mathsf{K}'/\mathsf{N}'$ is universally right localizable with respect to the localizing class $\mathcal{S}_{\mathsf{N}}$, associated with the null system N. We write $\mathbf{R}\mathrm{F}$ for the right localization of F. Ditto for left localizations.

I.6.4 DEFINITION
In the setting above, let A *be a full subcategory of* K. *Consider the following conditions.*

(I.30) For any $\mathrm{X} \in \mathsf{K}$, *there exists* $(\mathrm{X} \to \mathrm{I}) \in \mathcal{S}_{\mathsf{N}}$ *with* $\mathrm{I} \in \mathsf{A}$.

(I.31) For any $\mathrm{X} \in \mathsf{K}$, *there exists* $(\mathrm{P} \to \mathrm{X}) \in \mathcal{S}_{\mathsf{N}}$ *with* $\mathrm{P} \in \mathsf{A}$.

(I.32) $\mathrm{F}(\mathsf{N} \cap \mathsf{A}) \subset \mathsf{N}'$.

The subcategory A *is* F-injective *(with respect to* $(\mathsf{N}, \mathsf{N}')$ *of course) if it satisfies conditions (I.30) and (I.32) above. The subcategory* A *is* F-projective *if it satisfies conditions (I.31) and (I.32) above.*

Note that if $\mathrm{F}(\mathsf{N}) \subset \mathsf{N}'$ then the whole category K is both F-injective and F-projective.

I.6.5 THEOREM
If I *is* F-*injective then* F *is universally right localizable and its right localization* $\mathbf{R}\mathrm{F}$ *is a* Δ-*functor. Moreover* $\mathbf{R}\mathrm{F}$ *may be defined by the following diagram*

$$\begin{tikzcd}
& \mathsf{K} \arrow[r] & \mathsf{K}/\mathsf{N} \arrow[dd, dashed, "\mathbf{R}\mathrm{F}"] \\
\mathsf{I} \arrow[ur, hook] \arrow[r] \arrow[drr] & \mathsf{I}/(\mathsf{I} \cap \mathsf{N}) \arrow[ur, "\sim"] \arrow[dr] & \\
& & \mathsf{K}'/\mathsf{N}'
\end{tikzcd}$$

and

(I.33) $\mathbf{R}F(X) \simeq F(I)$, *for* $(X \to I) \in \mathcal{S}_N$ *and* $I \in \mathsf{I}$.

Similarly, if P *is* F*-projective then* F *is universally left localizable and its left localization* $\mathbf{L}F$ *is a* Δ*-functor. Moreover* $\mathbf{L}F$ *may be defined by the following diagram*

$$\begin{tikzcd}
& \mathsf{K} \arrow[rr] & & \mathsf{K}/\mathsf{N} \arrow[dd, dashed, "\mathbf{L}F"] \\
\mathsf{P} \arrow[ur, hook] \arrow[r] \arrow[drrr] & \mathsf{P}/(\mathsf{P}\cap\mathsf{N}) \arrow[urr, "\sim"] \arrow[drr] & & \\
& & & \mathsf{K}'/\mathsf{N}'
\end{tikzcd}$$

and

(I.34) $\mathbf{L}F(X) \simeq F(P)$, *for* $(P \to X) \in \mathcal{S}_N$ *and* $P \in \mathsf{P}$.

Proof
It is a straightforward consequence of Theorem I.6.3 applied to the functor $Q' \circ F$.

I.6.6 Theorem
Let $F : \mathsf{K} \to \mathsf{K}'$ *and* $G : \mathsf{K}' \to \mathsf{K}''$ *be two* Δ*-functors and let* N, N', N'' *be null systems in* K, K' *and* K'' *respectively (we consider localizations with respect to these null systems).*

- *If* $\mathbf{R}F$, $\mathbf{R}G$ *and* $\mathbf{R}(G \circ F)$ *exist then there is a canonical morphism of functors*

  (I.35) $$\mathbf{R}(G \circ F) \longrightarrow \mathbf{R}G \circ \mathbf{R}F.$$

- *Let* I *be an* F*-injective subcategory of* K *and let* I' *be a* G*-injective subcategory of* K'*. Assume that* $F(\mathsf{I}) \subset \mathsf{I}'$*. Then* I *is* $(G \circ F)$*-injective and* (I.35) *is an isomorphism.*

Dually:

- *If* $\mathbf{L}F$, $\mathbf{L}G$ *and* $\mathbf{L}(G \circ F)$ *exist then there is a canonical morphism of functors*

  (I.36) $$\mathbf{L}G \circ \mathbf{L}F \longrightarrow \mathbf{L}(G \circ F).$$

- *Let* P *be an* F*-projective subcategory of* K *and let* P' *be a* G*-projective subcategory of* K'*. Assume that* $F(\mathsf{P}) \subset \mathsf{P}'$*. Then* P *is* $(G \circ F)$*-projective and* (I.36) *is an isomorphism.*

Proof
Let's prove the first assertion. By Definition (I.24), for any functor $J : \mathsf{K}/\mathsf{N} \to \mathsf{K}''/\mathsf{N}''$ we have a bijection

$$\mathrm{Hom}\,(\mathbf{R}(G \circ F), J) \simeq \mathrm{Hom}\,(Q'' \circ G \circ F, J \circ Q)\,,$$

which for $J = \mathbf{R}G \circ \mathbf{R}F$ yields

$$\text{(I.37)} \quad \mathrm{Hom}\left(\mathbf{R}(G \circ F), \mathbf{R}G \circ \mathbf{R}F\right) \simeq \mathrm{Hom}\left(Q'' \circ G \circ F, \mathbf{R}G \circ \mathbf{R}F \circ Q\right).$$

Also by definition we have two natural transformations $Q'F \to \mathbf{R}F \circ Q$ and $Q''G \to \mathbf{R}G \to Q'$.

$$\begin{tikzcd} \mathsf{K} \arrow[rr, bend left, "Q' \circ F"] \arrow[rr, bend right, "\mathbf{R}F \circ Q"'] \arrow[rr, phantom, "\Downarrow"] & & \mathsf{K}'/\mathsf{N}' \end{tikzcd} \quad \begin{tikzcd} \mathsf{K}' \arrow[rr, bend left, "Q'' \circ G"] \arrow[rr, bend right, "\mathbf{R}G \circ Q'"'] \arrow[rr, phantom, "\Downarrow"] & & \mathsf{K}''/\mathsf{N}'' \end{tikzcd},$$

We thus deduce the morphisms $Q'' \circ G \circ F \to \mathbf{R}G \circ Q' \circ F \to \mathbf{R}G \circ \mathbf{R}F \circ Q$, which through the bijection (I.37) gives (I.35).

Let's prove the second assertion. The subcategory I is obviously $(G \circ F)$-injective. Applying THEOREM I.6.5 to the present case, we find how to compute localizations. Let $X \in \mathsf{K}$ and let $(X \to I) \in \mathcal{S}_{\mathsf{N}}$, where $I \in \mathsf{I}$. Then $\mathbf{R}F(X) \simeq F(I)$, in turn $F(I) \in \mathsf{I}'$ therefore $\mathbf{R}G(F(I)) = G(F(I))$. Finally, since I is $(G \circ F)$-injective, we have $\mathbf{R}(G \circ F)(X) \simeq G \circ F(I) = \mathbf{R}G(F(I)) \simeq \mathbf{R}G(\mathbf{R}F(X))$, for all $X \in \mathsf{K}$. The proof for projectives is similar. ❧

Derived Functors

Let's move on to derived categories. Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor between abelian categories and let the Δ-functors $\mathbf{K}^*(F) : \mathbf{K}^*(\mathcal{A}) \to \mathbf{K}^*(\mathcal{A}')$ be its extensions to the homotopy categories, for $* = \mathsf{ub}, +, -, \mathsf{b}$. We usually denote the extensions of F still by F. Let, as usual, $*$ be one of $\mathsf{ub}, +, -\mathsf{b}$.

We say that F is *right derivable* (or *admits a right derived functor*) on $\mathbf{K}^*(\mathcal{A})$ if the Δ-functor $\mathbf{K}^*(F)$ is universally right localizable (with respect to $\mathsf{N}^*(\mathcal{A})$ and $\mathsf{N}^*(\mathcal{A}')$).

In such a case, the right localization of F is denoted by $\mathbf{R}^*F$ and is called *the right* (total) *derived functor* of F. The functor $H^k \circ \mathbf{R}^*F$ is denoted by $\mathbf{R}^kF$ and called *the k-th* (classical, right) *derived functor* of F.

Similarly, F is *left derivable* if $\mathbf{K}^*(F)$ is universally left localizable. The left localization of F is denoted by $\mathbf{L}^*F$ and called *the left* (total) *derived functor* of F. Finally $H^k \circ \mathbf{L}^*F$ is denoted by $\mathbf{L}^kF$ and called *the k-th* (classical, left) *derived functor* of F.

I.6.7 Remark Notice that $\mathbf{R}^*F$ and $\mathbf{L}^*F$ (when they exist) are Δ-functors. Also $\mathbf{R}^kF$ and $\mathbf{L}^kF$ (when they exist of course) are cohomological functors from $\mathbf{D}^*(\mathcal{A})$ to $\mathcal{A}'$. When no confusion arises we drop the superscript $*$ from the notation and simply denote the derived functors by $\mathbf{R}F$ and $\mathbf{L}F$.

Injective and Projective Subcategories of an Abelian Category

We say that a full additive subcategory $\mathcal{I}$ of $\mathcal{A}$ is F-*injective* if the subcategory $\mathbf{K}^+(\mathcal{I})$ is $\mathbf{K}^+(F)$-injective, in the sense of DEFINITION I.6.4.

Similarly, $\mathcal{P}$ is F-*projective* if $\mathbf{K}^-(\mathcal{P})$ is $\mathbf{K}^-(F)$-projective.

I.6.8 Remark By definition $\mathcal{I}$ is F-injective if and only if

(I.38) for any $X^\cdot \in \mathbf{K}^+(\mathcal{A})$ there exists a qis $X^\cdot \to I^\cdot$, with $I^\cdot \in \mathbf{K}^+(\mathcal{I})$;

(I.39) for any exact complex $I^\cdot \in \mathbf{K}^+(\mathcal{I})$, $F(I^\cdot)$ is exact.

Analogously, $\mathcal{P}$ is F-projective if and only if

(I.40) for any $X^\cdot \in \mathbf{K}^-(\mathcal{A})$ there exists a qis $P^\cdot \to X^\cdot$, with $P^\cdot \in \mathbf{K}^-(\mathcal{P})$;

(I.41) for any exact complex $P^\cdot \in \mathbf{K}^-(\mathcal{P})$, $F(P^\cdot)$ is exact.

When dealing with injective and projective subcategories, deriving functors becomes a *simple* matter.

I.6.9 Theorem
If $\mathcal{I}$ is F-injective then $\mathbf{R}^+F : \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}^+(\mathcal{A}')$ exists and

(I.42) $\mathbf{R}^+F(X^\cdot) \simeq F(I^\cdot)$, *for any qis* $X^\cdot \to I^\cdot$, *with* $I^\cdot \in \mathbf{K}^+(\mathcal{I})$.

If $\mathcal{P}$ is F-projective then $\mathbf{L}^-F : \mathbf{D}^-(\mathcal{A}) \to \mathbf{D}^-(\mathcal{A}')$ exists and

(I.43) $\mathbf{L}^+F(X^\cdot) \simeq F(P^\cdot)$, *for any qis* $P^\cdot \to X^\cdot$, *with* $P^\cdot \in \mathbf{K}^+(\mathcal{P})$.

Proof
It follows from Theorem I.6.5 above.

Now we seek for a criterion to determine injectiveness and projectiveness of subcategories.

I.6.10 Theorem
Let $\mathcal{I}$ be a full additive subcategory of $\mathcal{A}$ and let F be left exact. Assume that

(I.44) $\mathcal{I}$ is cogenerating;

(I.45) for any exact sequence $0 \to I' \to I \to X'' \to 0$, *with* $I', I \in \mathcal{I}$, *we have* $X'' \in \mathcal{I}$;

(I.46) for any exact sequence $0 \to I' \to I \to I'' \to 0$, *with* $I', I, I'' \in \mathcal{I}$, $0 \to F(I') \to F(I) \to F(I'') \to 0$ *is exact.*

Then $\mathcal{I}$ is F-injective.
Similarly, let F be right exact. Assume that

(I.47) $\mathcal{P}$ is generating;

(I.48) for any exact sequence $0 \to X' \to P \to P'' \to 0$, *with* $P'', P \in \mathcal{P}$, *we have* $X' \in \mathcal{P}$;

(I.49) for any exact sequence $0 \to P' \to P \to P'' \to 0$, *with* $P', P, P'' \in \mathcal{P}$, $0 \to F(P') \to F(P) \to F(P'') \to 0$ *is exact.*

Then $\mathcal{P}$ is F-projective.

Proof
See [KS06, Corollary 13.3.8].

F-acyclic Objects

When the existence of the derived functor is already known we can find other injective and projective subcategories of $\mathcal{A}$.

I.6.11 Remark Let $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor between abelian categories and assume that there exists an F-injective subcategory of $\mathcal{A}$. One notices that $\mathbf{R}^kF(X) = 0$, for $X \in \mathcal{A}$ and $k < 0$, if F is furthermore left exact then $\mathbf{R}^0F(X) \simeq F(X)$. Indeed, for $X \in \mathcal{A}$ and $X \to I^\cdot$ a qis, the morphism $X \to I^\cdot \to \tau^{\leq 0}I^\cdot$ is a qis.

If F is right derivable, an object $X \in \mathcal{A}$ is called *right* F*-acyclic* if

$$\mathbf{R}^k F(X) = 0$$

for $k \neq 0$. If $\mathcal{I}$ is an F-injective subcategory of $\mathcal{A}$, then any object of $\mathcal{I}$ is right F-acyclic.

If $\mathcal{A}$ has enough injectives then the full subcategory $\mathcal{I}_{\mathcal{A}}$ of injectives is F-injective for any additive functor F. Indeed, any exact complex in $\mathbf{K}^+(\mathcal{I}_{\mathcal{A}})$ is homotopic to zero by Lemma I.5.5. In particular, $\mathbf{R}F$ exists.

Of course we have a mirror remark for left derived functors.

I.6.12 Remark Dually, assume that there exists an F-projective subcategory of $\mathcal{A}$. One notices that $\mathbf{L}^k F(X) = 0$, for $X \in \mathcal{A}$ and $k > 0$, if F is furthermore right exact then $\mathbf{L}^0 F(X) \simeq F(X)$.

If F is left derivable, an object $X \in \mathcal{A}$ is called *left* F*-acyclic* if $\mathbf{L}^k F(X) = 0$ for $k \neq 0$. If $\mathcal{P}$ is an F-projective subcategory of $\mathcal{A}$, then any object of $\mathcal{P}$ is left F-acyclic.

If $\mathcal{A}$ has enough projectives then the full subcategory $\mathcal{P}_{\mathcal{A}}$ projectives is F-projective for any additive functor F. In particular, $\mathbf{R}F$ exists.

I.6.13 Theorem
Let F *be left exact and let* $\mathcal{I}$ *be* F*-injective. Write* $\mathcal{I}_F$ *for the full subcategory of* $\mathcal{A}$ *consisting of right* F*-acyclic objects. Then* $\mathcal{I}_F$ *contains* $\mathcal{I}$ *and satisfies conditions* (I.44)-(I.46) *above. In particular* $\mathcal{I}_F$ *is* F*-injective.*

Dually, let F *be right exact and* $\mathcal{P}$ F*-projective. Then* $\mathcal{P}_F$*, the category of left* F*-acyclics, contains* $\mathcal{P}$ *and satisfies* (I.47)-(I.49) *above.*

Proof
We want to prove that $\mathcal{I}_F$ satisfies the hypotheses of Theorem I.6.10. Since any object in $\mathcal{I}$ is F-acyclic we have that $\mathcal{I}_F$ contains I, and therefore is cogenerating. Let $0 \to X' \to X \to X'' \to 0$ be exact, where X' and X are F-acyclic. Then, regarding the above sequence as a d.t. in $\mathbf{D}(\mathcal{A})$ and since $\mathbf{R}^k F$ exists and is cohomological, we have a long exact sequence

$$\cdots \to \mathbf{R}^j F(X) \to \mathbf{R}^j F(X'') \to \mathbf{R}^{j+1} F(X') \cdots$$

which implies that $\mathbf{R}^j F(X'') = 0$ for $k \neq 0$, and hence that X is right F-acyclic. By Remark I.6.11, the above exact sequence yields an exact sequence

$$0 \to F(X') \to F(X) \to F(X'') \to \mathbf{R}^1 F(X') = 0$$

therefore $\mathcal{I}_F$ is F-injective. For projectives the proof is analogous. ∾

We finish with the composition of derived functors. *Composition of Derived Functors*

I.6.14 Theorem
Let $F : \mathcal{A} \to \mathcal{B}$ *and* $G : \mathcal{B} \to \mathcal{C}$ *be two additive functors among abelian categories. Assume that the right derived functors* $\mathbf{R}F$ *and* $\mathbf{R}G$ *exist on* $\mathbf{D}^+$*. Then there is a canonical morphism of functors*

$$\mathbf{R}(G \circ F) \longrightarrow \mathbf{R}G \circ \mathbf{R}F. \tag{I.50}$$

Assume moreover that there exist an F*-injective subcategory* $\mathcal{I}$ *and a* G*-injective subcategory* $\mathcal{J}$ *such that* $F(\mathcal{I}) \subset \mathcal{J}$*. Then* $\mathcal{I}$ *is* $(G \circ F)$*-injective and the morphism* (I.50) *induces an isomorphism*

$$\mathbf{R}(G \circ F) \xrightarrow{\sim} \mathbf{R}G \circ \mathbf{R}F. \tag{I.51}$$

Dually, assume that the left derived functors $\mathbf{L}F$ *and* $\mathbf{L}G$ *exist on* $\mathbf{D}^-$*. Then there is a canonical morphism of functors*

$$\mathbf{L}G \circ \mathbf{L}F \longrightarrow \mathbf{L}(G \circ F). \tag{I.52}$$

Assume moreover that there exist an F*-projective subcategory* $\mathcal{P}$ *and a* G*-projective subcategory* $\mathcal{Q}$ *such that* $F(\mathcal{P}) \subset \mathcal{Q}$*. Then* $\mathcal{P}$ *is* $(G \circ F)$*-projective and the morphism* (I.50) *induces an isomorphism*

$$\mathbf{L}G \circ \mathbf{L}F \xrightarrow{\sim} \mathbf{L}(G \circ F). \tag{I.53}$$

Proof
All assertions are just an application of Theorem I.6.6 to the functors $\mathbf{K}^+(F)$, $\mathbf{K}^+(G)$, $\mathbf{K}^-(F)$ and $\mathbf{K}^-(G)$. ❧

I.7 BIFUNCTORS

Double Complexes

A *double complex* (or *bicomplex*) $X^{\cdot,\cdot}$, in an additive category $\mathcal{A}$, is a collection of objects $(X^{p,q})_{p,q\in\mathbb{Z}}$ and differentials

$$\partial^{p,q}_{X^{\cdot,\cdot}} : X^{p,q} \longrightarrow X^{p+1,q}$$
$$\delta^{p,q}_{X^{\cdot,\cdot}} : X^{p,q} \longrightarrow X^{p,q+1}$$

such that $\partial^{p+1,q}\partial^{p,q} = 0$, $\delta^{p,q+1}\delta^{p,q} = 0$ and $\delta^{p+1,q}\partial^{p,q} = \partial^{p,q+1}\delta^{p,q}$. A double complexe can be visualized as a commutative diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & X^{p,q+1} \arrow[u] \arrow[r, "\partial^{p,q+1}"] & X^{p+1,q+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & X^{p,q} \arrow[u, "\delta^{p,q}"] \arrow[r, "\partial^{p,q}"] & X^{p+1,q} \arrow[u, "\delta^{p,q+1}"] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

where the rows and the columns are complexes. Often, for a double complex $X^{\cdot,\cdot}$, the horizontal differentials $\partial^{p,q}_X$ are denoted by ${d'}^{p,q}_X$ and the vertical differentials $\delta^{p,q}_X$ are denoted by ${d''}^{p,q}_X$. Morphisms of double complexes are defined as collection of morphisms $(f^{p,q})_{p,q\in\mathbb{Z}}$ commuting with all the differentials. We have thus constructed the additive category $\mathbf{C}^2(\mathcal{A})$ of double complexes in $\mathcal{A}$.

We define two functors $F_I, F_{II} : \mathbf{C}^2(\mathcal{A}) \to \mathbf{C}(\mathbf{C}(\mathcal{A}))$. The first considers the columns of a double complex as objects in $\mathbf{C}(\mathcal{A})$ and the horizontal maps as differentials between them. The second functor does the same for rows. Namely, if $X^{\cdot,\cdot}$ is a double complex, then $F_I(X^{\cdot,\cdot}) = X_I$ is the complex (of complexes) with components

$$F_I(X^{\cdot,\cdot})^n = X_I^n = X^{n,\cdot}$$

and $F_{II}(X^{\cdot,\cdot}) = X_{II}$ is the complex with components

$$F_{II}(X^{\cdot,\cdot})^m = X_{II}^m = X^{\cdot,m}.$$

It is straightforward to check that the functors F_I and F_{II} are equivalences of categories.

Assume now that $\mathcal{A}$ admits countable coproducts, or assume that the complexes we work with satisfy the following finiteness condition *Total Complex*

(I.54) for any $n \in \mathbb{Z}$, the set $\{(p,q) \in \mathbb{Z} \times \mathbb{Z} \mid p+q = n, X^{p,q} \neq 0\}$ is finite,

which is always satisfied if we work with *first quadrant* or *third quadrant* double complexes (i.e. $X^{p,q} = 0$ outside the first or third quadrant). We now define a functor $\mathrm{tot} = \mathrm{tot}_{\oplus} : \mathbf{C}^2(\mathcal{A}) \to \mathbf{C}(\mathcal{A})$, which takes a double complex and gives a (*simple*) complex. If $(X^{\cdot,\cdot}, \partial_X, \delta_X)$ is a double complex then $(\mathrm{tot}(X^{\cdot,\cdot})^n, d^n)_n$ is called the *total complex* (or *simple complex*) associated with $X^{\cdot,\cdot}$. Its components are

$$\mathrm{tot}(X^{\cdot,\cdot})^n = \bigoplus_{p+q=n} X^{p,q}$$

and the differentials d^n are uniquely determined by their restrictions

$$d^n|_{X^{p,q}} = \delta^{p+1,q}\partial^{p,q} + (-1)^p \partial^{p,q+1}\delta^{p,q}$$

on each $X^{p,q}$ such that $p+q=n$.

By *bifunctor* we mean simply a functor $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$ defined on the product of two categories. A bifunctor is said to be additive, left exact, right exact, exact, coholomogical or a Δ-bifunctor if it so with respect to each variable. *Bifunctors*

I.7.1 Example For a $\Bbbk$-linear abelian category $\mathcal{C}$, the functor $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^\circ \times \mathcal{C} \to \Bbbk\text{-}\mathsf{Mod}$ is a left exact bifunctor.

Let $F : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$ be an additive bifunctor between additive categories. We want to extend F to complexes. Given two complexes $X^\cdot, Y^\cdot$ we form the double complex $F^{\cdot,\cdot}(X^\cdot, Y^\cdot)$ with components $F^{p,q}(X^\cdot, Y^\cdot) = F(X^p, Y^q)$ and with horizontal differential $\partial^{p,q} = F(d_X^p, Y^q)$ and vertical differential $\delta^{p,q} = F(X^p, d_Y^q)$. Composing with the total complex functor we obtain the desired extension $F^\cdot$ of F to complexes. Since we want to use the finiteness condition (I.54), we impose some boundedness conditions: we define $F^\cdot$ (which we usually denote simply by F) to be

$$F^\cdot = \mathrm{tot} \circ F : \mathbf{C}^*(\mathcal{A}) \times \mathbf{C}^*(\mathcal{A}') \longrightarrow \mathbf{C}^*(\mathcal{A}'')$$

where $* = +, -, \mathsf{b}$.

One notices (see [KS06, Proposition 11.6.4]) that the extensions of F are compatible with homotopy, thus obtaining functors

$$F : \mathbf{K}^*(\mathcal{A}) \times \mathbf{K}^*(\mathcal{A}') \longrightarrow \mathbf{K}^*(\mathcal{A}'')$$

for $* = +, -, \mathrm{b}$. Before passing onto the derived category we take a look at a more general situation. We study the localization of a functor of two variables.

Localization of Bifunctors

I.7.2 Remark If C and C' are two categories and $\mathcal{S}$ and $\mathcal{S}'$ are two multiplicative systems then $\mathcal{S} \times \mathcal{S}'$ is a multiplicative system in $\mathsf{C} \times \mathsf{C}'$ and $(\mathsf{C} \times \mathsf{C}')_{\mathcal{S}\times\mathcal{S}'}$ is equivalent to $\mathsf{C}_{\mathcal{S}} \times \mathsf{C}'_{\mathcal{S}'}$.

Let K, K', K'' be three triangulated categories with respective null systems N, N',N'' and respective quotient functors Q, Q', Q''. Let $F : \mathsf{K}\times\mathsf{K}' \to \mathsf{K}''$ be a Δ-bifunctor. We say that F is *right localizable* (with respect to $(\mathsf{N} \times \mathsf{N}', \mathsf{N}'')$) if the functor $Q'' \circ F$ is universally right localizable with respect to the multiplicative system $\mathcal{S}_{\mathsf{N}} \times \mathcal{S}_{\mathsf{N}'}$ (see remark above). Similarly we say that F is *left localizable* if $Q'' \circ F$ is universally left localizable. Again we omit the dependencies from the null systems in our notations. We now want to define injective and projective subcategories in the above setting.

Let I, I' be full subcategories of K, K'. We say that the pair $(\mathsf{I}, \mathsf{I}')$ is F-*injective* if the following conditions hold.

(I.55) For any $I \in \mathsf{I}$, I' is $F(I, \text{-})$-injective.

(I.56) For any $I' \in \mathsf{I}'$, I is $F(\text{-}, I')$-injective.

Unraveling the definitions we see that $(\mathsf{I}, \mathsf{I}')$ is F-injective if and only if the following conditions are satisfied.

(I.57) For any $X \in \mathsf{K}$, there exists $(X \to I) \in \mathcal{S}_{\mathsf{N}}$ with $I \in \mathsf{I}$.

(I.58) For any $X' \in \mathsf{K}'$, there exists $(X' \to I') \in \mathcal{S}_{\mathsf{N}'}$ with $I' \in \mathsf{I}'$.

(I.59) For any $I \in \mathsf{I}$, $I' \in \mathsf{I}'$, we have $F(I, I') \in \mathsf{N}''$ if $I \in \mathsf{N}$ or $I' \in \mathsf{N}'$. (viz. $F(\mathsf{I} \cap \mathsf{N}, \mathsf{I}') \subset \mathsf{N}''$ and $F(\mathsf{I}, \mathsf{I}' \cap \mathsf{N}') \subset \mathsf{N}''$)

Similarly, if P, P' are full subcategories of K, K', we say that the pair $(\mathsf{P}, \mathsf{P}')$ is F-*projective* if the following conditions hold.

(I.60) For any $P \in \mathsf{P}$, P' is $F(P, \text{-})$-projective.

(I.61) For any $P' \in \mathsf{P}'$, P is $F(\text{-}, I')$-projective.

Of course, the above conditions are equivalent to the conditions below.

(I.62) For any $X \in \mathsf{K}$, there exists $(P \to X) \in \mathcal{S}_{\mathsf{N}}$ with $P \in \mathsf{P}$.

(I.63) For any $X' \in \mathsf{K}'$, there exists $(P' \to X') \in \mathcal{S}_{\mathsf{N}'}$ with $P' \in \mathsf{P}'$.

(I.64) For any $P \in \mathsf{P}$, $P' \in \mathsf{P}'$, we have $F(P, P') \in \mathsf{N}''$ if $P \in \mathsf{N}$ or $P' \in \mathsf{N}'$. (viz. $F(\mathsf{P} \cap \mathsf{N}, \mathsf{P}') \subset \mathsf{N}''$ and $F(\mathsf{P}, \mathsf{P}' \cap \mathsf{N}') \subset \mathsf{N}''$)

The following results are a consequence of the theory of localizations of functors developed so far.

I.7.3 THEOREM
In the setting above, assume the pair $(\mathsf{I}, \mathsf{I}')$ *to be* F*-injective. Then* F *is right localizable, its right localization* $\mathbf{R}\mathrm{F} : \mathsf{K}/\mathsf{N} \times \mathsf{K}'/\mathsf{N}' \to \mathsf{K}''/\mathsf{N}''$ *is a* Δ*-bifunctor and*

(I.65) $\mathbf{R}\mathrm{F}(\mathrm{X}, \mathrm{X}') \simeq \mathrm{F}(\mathrm{I}, \mathrm{I}')$, *for* $(\mathrm{X} \to \mathrm{I}) \in \mathcal{S}_{\mathsf{N}}$, $(\mathrm{X}' \to \mathrm{I}') \in \mathcal{S}_{\mathsf{N}'}$, $\mathrm{I} \in \mathsf{I}$, $\mathrm{I}' \in \mathsf{I}'$.

Similarly, assume the pair $(\mathsf{P}, \mathsf{P}')$ *to be* F*-projective. Then* F *is left localizable, its left localization* $\mathbf{L}\mathrm{F} : \mathsf{K}/\mathsf{N} \times \mathsf{K}'/\mathsf{N}' \to \mathsf{K}''/\mathsf{N}''$ *is a* Δ*-bifunctor and*

(I.66) $\mathbf{L}\mathrm{F}(\mathrm{X}, \mathrm{X}') \simeq \mathrm{F}(\mathrm{P}, \mathrm{P}')$, *for* $(\mathrm{P} \to \mathrm{X}) \in \mathcal{S}_{\mathsf{N}}$, $(\mathrm{P}' \to \mathrm{X}') \in \mathcal{S}_{\mathsf{N}'}$, $\mathrm{P} \in \mathsf{P}$, $\mathrm{P}' \in \mathsf{P}'$.

I.7.4 COROLLARY
In the above setting assume that

(I.67) $\mathrm{F}(\mathsf{I}, \mathsf{N}') \subset \mathsf{N}''$;

(I.68) for any $\mathrm{X}' \in \mathsf{K}'$, I *is* $\mathrm{F}(\text{-}, \mathrm{X}')$*-injective.*

Then F *is right localizable and*

$$\mathbf{R}\mathrm{F}(\mathrm{X}, \mathrm{X}') \simeq \mathbf{R}\mathrm{F}(\text{-}, \mathrm{X}')(\mathrm{X}). \tag{I.69}$$

Analogously, assume that

(I.70) $\mathrm{F}(\mathsf{P}, \mathsf{N}') \subset \mathsf{N}''$;

(I.71) for any $\mathrm{X}' \in \mathsf{K}'$, P *is* $\mathrm{F}(\text{-}, \mathrm{X}')$*-projective.*

Then F *is left localizable and*

$$\mathbf{L}\mathrm{F}(\mathrm{X}, \mathrm{X}') \simeq \mathbf{L}\mathrm{F}(\text{-}, \mathrm{X}')(\mathrm{X}). \tag{I.72}$$

Of course a similar statement holds when switching the variables of F around.

The definitions and results for derived categories are analogous to the ones for triangulated categories. We just state an additional result, a proof can be found in [KS06, COROLLARY 13.4.5]. *Derived Bifunctors*

I.7.5 COROLLARY
Let $\mathcal{I}$ *be a cogenerating subcategory of an abelian category* $\mathcal{A}$*. Let* $\mathrm{F} : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''$ *be an additive bifunctor. Assume that*

(I.73) for any $\mathrm{I} \in \mathcal{I}$, $\mathrm{F}(\mathrm{I}, \text{-})$ *is exact;*

(I.74) for any $\mathrm{X}' \in \mathcal{A}$, $\mathcal{I}$ *is* $\mathrm{F}(\text{-}, \mathrm{X}')$*-injective.*

Then F *is right derivable and for* $\mathrm{X}^\cdot \in \mathbf{K}^+(\mathcal{A}), \mathrm{X}'^\cdot \in \mathbf{K}^+(\mathcal{A}')$

(I.75) $\mathbf{R}\mathrm{F}(\mathrm{X}^\cdot, \mathrm{X}'^\cdot) \simeq \mathrm{Q}'' \circ \mathbf{K}\mathrm{F}(\mathrm{I}^\cdot, \mathrm{X}'^\cdot)$, *for a qis* $(\mathrm{X}^\cdot \to \mathrm{I}^\cdot)$ *with* $\mathrm{I}^\cdot \in \mathbf{K}^+(\mathcal{I})$.

In particular for $\mathrm{X} \in \mathcal{A}$ *and* $\mathrm{X}' \in \mathcal{A}$, $\mathbf{R}\mathrm{F}(\mathrm{X}, \mathrm{X}') = \mathbf{R}\mathrm{F}(\text{-}, \mathrm{X}')(\mathrm{X})$.

There is of course a similar statement for projectives.

Applying these results to the functor $\mathrm{Hom}_{\mathcal{A}}$ one proves that (see [KS06] for details) $\mathrm{Ext}^k_{\mathcal{A}}(\mathrm{X}, \mathrm{Y}) = \mathbf{R}^k \mathrm{Hom}_{\mathcal{A}}(\mathrm{X}, \mathrm{Y}) \simeq \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(\mathrm{X}, \mathrm{Y}[k])$.

I.8 SPECTRAL SEQUENCES

We briefly recall a few results about spectral sequences. For proofs one may consult [GM03] and [Wei94]. For a more careful treatment we suggest [Mur06], which follows [Gro67, 0_{III}-§11].

Filtrations

First we give a few definitions. Let's fix an abelian category $\mathcal{A}$. Recall that a *subobject* of an object X is simply a monomorphism $W \hookrightarrow X$. We define a relation on subobjects. Given two subobjects of X, $\alpha_1 : X_1 \hookrightarrow X$, $\alpha_2 : X_2 \hookrightarrow X$, we say that α_1 *precedes* α_2, and often write $X_1 \subset X_2$, if the first factors through the second

$$\begin{tikzcd} X_1 \arrow[dd, dashed] \arrow[dr, hook, "\alpha_1"] & \\ & X \\ X_2 \arrow[ur, hook, "\alpha_2"'] & \end{tikzcd}$$

and we also say that α_2 *follows* α_1. We say that two subobjects are equivalent if they mutually precede each other. Given a family of subobjects $\{\alpha_i : X_i \hookrightarrow X\}_i$ we define an *intersection* of the family as a subobject $\alpha : W \hookrightarrow X$ such that

(I.76) α precedes all α_i;

(I.77) for any other subobject $\alpha' : W' \hookrightarrow X$, preceding all α_i, α' precedes α.

Any two intersections are equivalent (in the sense above). We often write $\bigcap_i X_i$ for the intersection of the α_is. Dually we define a *union* of the α_is as a subobject $\beta : Y \hookrightarrow X$ such that

(I.78) β follows all α_i;

(I.79) any other subobject following all α_i follows β.

Any two unions are equivalent. We often write $\bigcup_i X_i$ for the union of the α_is.

A (decreasing) *filtration* of an object X of $\mathcal{A}$ is a sequence of subobjects of X

$$\cdots \supset F^0X \supset F^1X \supset \cdots F^pX \supset \cdots$$

and we say that it is *regular* if $\bigcap_p F^pX = 0$ and $\bigcup_p F^pX = X$. We say that it is *finite* if it is stationary on both sides: i.e. there exist p_0 and p_1 such that $F^pX = F^{p_0}X$, for all $p \leq p_0$, and $F^pX = F^{p_1}X$, for all $p \geq p_1$.

$$\cdots = F^{p_0-1}X = F^{p_0}X \supset \cdots \supset F^{p_1}X = F^{p_1+1} = \cdots$$

Spectral Sequences

A *spectral sequence* $E = (E_r^{p,q}, E^n)$ (*starting on page* r_0) is given by the following pieces of data.

(I.80) For every $r \geq r_0$, there is given a collection E_r (called *the r-th page*) of objects $E_r^{p,q} \in \mathcal{A}$, where $p, q \in \mathbb{Z}$.

(I.81) There are given morphisms $d_r^{p,q} : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$, such that

$$d_r^{p+r,q-r+1} d_r^{p,q} = 0,$$

where $p, q \in \mathbb{Z}$.

(I.82) If we denote $\mathrm{H}^{p,q}(\mathrm{E}_r) = \mathrm{Ker}\, d_r^{p,q} / \mathrm{Im}\, d_r^{p-r,q+r-1}$ (the zeroth cohomology of the complex $\mathrm{E}_r^{p+\cdot,q-\cdot+1}$), there are given isomorphisms

$$\alpha_r^{p,q} : \mathrm{H}^{p,q}(\mathrm{E}_r) \to \mathrm{E}_{r+1}^{p,q}.$$

(I.83) For any p, q, there exists $r^* = r^*(p,q)$ such that, for any $r \geq r^*$, the differentials $d_r^{p-r,q+r-1}$ and $d_r^{p,q}$ vanish. In this[5] case, the morphisms $\alpha_r^{p,q}$, identify all $\mathrm{E}_r^{p,q}$, for $r \geq r^*$.

$$\mathrm{E}_{r^*}^{p,q} \simeq \mathrm{E}_{r^*+1}^{p,q} \simeq \mathrm{E}_{r^*+2}^{p,q} \simeq \cdots$$

We denote this object by $\mathrm{E}_\infty^{p,q}$.

(I.84) There are given a decreasing regular[6] filtration

$$\cdots \supset \mathrm{F}^p\mathrm{E}^n \supset \mathrm{F}^{p+1}\mathrm{E}^n \supset \cdots$$

on each E^n and isomorphisms

$$\beta^{p,q} : \mathrm{E}_\infty^{p,q} \to \mathrm{F}^p\mathrm{E}^{p+q}/\mathrm{F}^{p+1}\mathrm{E}^{p+q}.$$

We say that the spectral sequence E *converges* to (E^n), or that (E^n) is the *limit* of E, and write

$$\mathrm{E}_{r_0}^{p,q} \Longrightarrow \mathrm{E}^{p+q}.$$

A way to think of the information extracted out of a spectral sequence

$$\mathrm{E}_{r_0}^{p,n-p} \Longrightarrow \mathrm{E}^n,$$

is that the objects E^n are built up of *extensions* of the objects $\mathrm{E}_\infty^{p,n-p}$, which are cohomologies of cohomologies of cohomologies of the complexes $\mathrm{E}_r^{\cdot,\cdot}$.

We can form the additive category of spectral sequences by defining morphisms $f : (\mathrm{E}_r^{p,q}, \mathrm{E}^n) \to (\mathrm{E}_r'^{p,q}, \mathrm{E}'^n)$ as collections of maps $f_r^{p,q} : \mathrm{E}_r^{p,q} \to \mathrm{E}_r'^{p,q}$, $f^n : \mathrm{E}^n \to \mathrm{E}'^n$, commuting with the structural morphisms and compatible with filtrations.

I.8.1 Remark We notice that condition (I.83) is satisfied whenever the starting page E_{r_0} has vanishing objects outside an area of the form $p \geq p_0, q \geq q_0$ (or equivalently $p \leq p_0, q \leq q_0$). In fact, if an object $\mathrm{E}_r^{p,q} = 0$ vanishes then all objects $\mathrm{E}_{r+k}^{p,q} = 0$ vanish for $k \geq 0$. In this case, if we fix p and q, as the page number increases we are bound to reach an $r \geq r_0$ such that the differential $d_r^{p-r,q+r-1}$ starts from a zero object and $d_r^{p,q}$ ends on a zero object, hence they both vanish.

Moreover, in this particular case, we are assured that the filtrations $\{\mathrm{F}^p\mathrm{E}^n\}_p$ are all finite.

5 The actual definition of a spectral sequence does not require the vanishing of any differential, but introduces additional filtrations. Since it is common that a spectral sequence satisfies condition (I.83), we prefer to give this simplified definition (as done in [GM03]).

6 In the general definition the regularity assumption of the filtrations is not present either.

We say that a spectral sequence E *degenerates* at page r^* if all differentials $d_r^{p,q}$ vanish, for $r \geq r^*$. In such a case, $\mathrm{E}_\infty^{p,q} = \mathrm{E}_{r^*}^{p,q}$ for all p, q.

I.8.2 Theorem (The Grothendieck Spectral Sequence)
Let $\mathrm{F} : \mathcal{A} \to \mathcal{B}$, $\mathrm{G} : \mathcal{B} \to \mathcal{C}$ *be two left exact functors and let* $\mathcal{J}$ *be a* $\mathbf{K}^+(\mathrm{G})$*-injective subcategory. Assume* $\mathcal{A}$ *and* $\mathcal{B}$ *have enough injectives and assume that* $\mathrm{F}(\mathcal{I}_{\mathcal{A}}) \subset \mathcal{J}$*, viz. any bounded below complex of injectives maps to an element of* $\mathcal{J}$*. Then, for any complex* $\mathrm{A}^{\cdot} \in \mathbf{D}^+(\mathcal{A})$ *there exists a spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathbf{R}^p\mathrm{G}(\mathbf{R}^q\mathrm{F}(\mathrm{A}^{\cdot})) \Longrightarrow \mathbf{R}^{p+q}(\mathrm{G} \circ \mathrm{F})(\mathrm{A}^{\cdot}). \tag{I.85}$$

As a very special case we have the following useful spectral sequence.

I.8.3 Corollary
Let $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ *be a left exact functor and let* $\mathcal{A}$ *have enough injectives. Then, for any* $\mathrm{A}^{\cdot} \in \mathbf{D}^+(\mathcal{A})$*, there exists a spectral sequence*

$$\mathrm{E}_2^{p,q} = \mathbf{R}^p\mathrm{F}(\mathrm{H}^q(\mathrm{A}^{\cdot})) \Longrightarrow \mathbf{R}^{p+q}\mathrm{F}(\mathrm{A}^{\cdot}). \tag{I.86}$$

Proof
It is precisely the spectral sequence (I.85), in which $\mathrm{F} = \mathrm{Id}_{\mathcal{A}}$ and we relabel $\mathrm{G} = \mathrm{F}$.

Another useful spectral sequence is the following (see [Huy06, Remark 2.67]):

$$\mathrm{E}_1^{p,q} = \mathbf{R}^q\mathrm{F}(\mathrm{A}^p) \Longrightarrow \mathbf{R}^{p+q}\mathrm{F}(\mathrm{A}^{\cdot}), \tag{I.87}$$

where $\mathrm{F} : \mathcal{A} \to \mathcal{B}$ is left exact, $\mathcal{A}$ has enough injectives and $\mathrm{A}^{\cdot} \in \mathbf{D}^+(\mathcal{A})$.

I.8.4 Theorem
Let $\mathrm{F} : \mathbf{K}^+\mathcal{A} \to \mathbf{K}^+\mathcal{B}$ *be a* Δ*-functor admitting a right localization* $\mathbf{R}\mathrm{F} : \mathbf{D}^+\mathcal{A} \to \mathbf{D}^+\mathcal{B}$*. Assume* $\mathcal{A}$ *has enough injectives.*

(I.88) Suppose $\mathcal{C}$ *is a thick subcategory of* $\mathcal{B}$*,* $\mathbf{R}^i\mathrm{F}(\mathrm{A}) \in \mathcal{C}$ *for all* $i \in \mathbb{Z}$ *and* $\mathrm{A} \in \mathcal{A}$*, and that there exists an* $n \in \mathbb{Z}$ *such that* $\mathbf{R}^j\mathrm{F}(\mathrm{A}) = 0$ *for all* $j < n$ *and* $\mathrm{A} \in \mathcal{A}$*. Then* $\mathbf{R}\mathrm{F}$ *factors through* $\mathbf{D}^+_{\mathcal{C}}\mathcal{B}$*.*

$$\begin{tikzcd}
\mathbf{D}^+\mathcal{A} \arrow[rr, "\mathbf{R}\mathrm{F}"] \arrow[dr, dashed] & & \mathbf{D}^+\mathcal{B} \\
& \mathbf{D}^+_{\mathcal{C}}\mathcal{B} \arrow[ur, hook] &
\end{tikzcd}$$

(I.89) Suppose $\mathbf{R}\mathrm{F}(\mathrm{A}) \in \mathbf{D}^{\mathrm{b}}\mathcal{B}$ *for all* $\mathrm{A} \in \mathcal{A}$*, then* $\mathbf{R}\mathrm{F}$ *factors through* $\mathbf{D}^{\mathrm{b}}\mathcal{B}$*.*

$$\begin{tikzcd}
\mathbf{D}^{\mathrm{b}}\mathcal{A} \arrow[rr, "\mathbf{R}\mathrm{F}"] \arrow[dr, dashed] & & \mathbf{D}^+\mathcal{B} \\
& \mathbf{D}^{\mathrm{b}}\mathcal{B} \arrow[ur, hook] &
\end{tikzcd}$$

Proof

Both follow from the spectral sequence (I.86). The first assertion is a consequence of the thickness of $\mathcal{C}$, which by definition implies that $\mathcal{C}$ is abelian and closed under extensions. Let $X^\cdot \in \mathbf{D}^+\mathcal{A}$ be a bounded below complex. We need to show that $H^n(\mathbf{R}F(X^\cdot)) = \mathbf{R}^n F(X^\cdot) \in \mathcal{C}$. Consider the spectral sequence $\mathbf{R}^p(H^q(X^\cdot)) \Rightarrow \mathbf{R}^{p+q}(X^\cdot)$. Now, $H^q(X^\cdot)$ is an object of $\mathcal{A}$ and by hypothesis $E_2^{p,q} = \mathbf{R}^p F(H^q(X^\cdot)) \in \mathcal{C}$, we have that $E_r^{p,q} \in \mathcal{C}$, for all $r \geq 2$, since $\mathcal{C}$ is closed under coholomogy of its complexes. Hence $E_\infty^{p,q} \in \mathcal{C}$, and we recall that the $E_\infty^{p,q} \simeq F^p E^{p+q}/F^{p+1}E^{p+q}$, where $F^p E^n$ is the filtration given by the spectral sequence. By our hypotheses and by REMARK I.8.1 the filtration of the limit $E^{p+q} = \mathbf{R}^{p+q}(X^\cdot)$ is finite, say

$$E^{p+q} = F^h E^{p+q} \supset \cdots \supset F^{h+k} E^{p+q} = 0,$$

and there exact sequences

$$0 \to F^p E^{p+q} \to F^{p+1} E^{p+q} \to F^{p+1} E^{p+q}/F^{p+2} E^{p+q} \to 0.$$

We claim that all $F^p E^{p+q}$ belong to $\mathcal{C}$. But this is simply a consequence of the finiteness of the filtration (which starts from $0 \in \mathcal{C}$) and of the thickness of $\mathcal{C}$ and of the fact that the quotients $F^{p+1}E^{p+q}/F^{p+2}E^{p+q}$ already belong to $\mathcal{C}$. Hence, $\mathbf{R}^{p+q}(X^\cdot) = E^{p+q} = F^h E^{p+q} \in \mathcal{C}$, thus proving our claim. For the second assertion one proceeds similarly, showing that for any $X^\cdot \in \mathbf{D}^b\mathcal{A}$ the complex $\mathbf{R}F(X^\cdot)$ has bounded cohomology, i.e. $\mathbf{R}^n F(X^\cdot)$ vanishes for large $|n|$. Again this is a consequence of the spectral sequence (I.86).

II

DERIVED FUNCTORS IN ALGEBRAIC GEOMETRY

Contents

In this chapter we present the essential results from algebraic geometry that are needed for the sequel. We omit some of the proofs and refer to the literature. As a reference we strongly advise [Huy06], but also the classic [Har66]. For the more standard material (viz. not involving derived categories) we refer to [Liu02] and of course [Har77].

II.1 COHERENT SHEAVES

Notation

To avoid confusion with sheaf coholomogy $H^\cdot(X,-)$ we use the notation below. Given a complex of sheaves $\mathscr{F}^\cdot \in \mathbf{C}(\mathcal{O}_X\text{-}\mathsf{Mod})$, on a ringed space X, we write $\mathscr{H}^i(\mathscr{F}^\cdot)$ for the i-th cohomology of the complex $\mathscr{F}^\cdot$ (which is again a sheaf).

We recall a few important facts about schemes that we need.

II.1.1 Big Theorem
Let $(X, \mathcal{O}_X)$ be a noetherian scheme. The following are true.

1. *An $\mathcal{O}_X$-module $\mathscr{F}$ is quasi-coherent if and only if for every open affine subset $U \subset X$ of X, $\mathscr{F}(U)^\sim \simeq \mathscr{F}|U$. The same holds for a coherent sheaf $\mathscr{F}$ if we add the condition of $\mathscr{F}|U$ being finitely generated.*
2. *A direct sum of quasi-coherent $\mathcal{O}_X$-modules is quasi-coherent.*
3. *A finite direct sum of coherent $\mathcal{O}_X$-modules is coherent.*
4. *If $\mathscr{F}$ and $\mathscr{G}$ are (quasi-)coherent $\mathcal{O}_X$-modules then so is $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$.*
5. *If $\mathscr{F}$ is coherent and $\mathscr{G}$ is (quasi-)coherent then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is (quasi-)coherent.*
6. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of (quasi-)coherent $\mathcal{O}_X$-modules then $\operatorname{Ker}\varphi$, $\operatorname{Coker}\varphi$, $\operatorname{Im}\varphi$ and $\operatorname{Coim}\varphi$, are (quasi-)coherent.*
7. *Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of $\mathcal{O}_X$-modules. If two of them are (quasi-)coherent then so is the third.*
8. $\mathsf{Qcoh}X$ *and* $\mathsf{Coh}X$ *are abelian subcategories of* $\mathcal{O}_X$-Mod. *Furthermore:*
   - *The category* $\mathsf{Qcoh}X$ *is a thick subcategory of* $\mathcal{O}_X$-Mod, *the category of sheaves of $\mathcal{O}_X$-modules;*

- *The category* $\mathsf{Coh}X$ *is a thick subcategory of* $\mathsf{Qcoh}X$.

9. *The category* $\mathcal{O}_X$-Mod *has enough injectives.*

10. *The category* $\mathsf{Qcoh}X$ *has enough injectives. More precisely, every quasi-coherent sheaf* $\mathscr{F}$ *admits an injection in a quasi-coherent sheaf* $\mathscr{I}$*, which is injective as an* $\mathcal{O}_X$*-module.*

Proof
For 1. through 7. see [Liu02, THEOREMS 5.1.7, 5.1.11, 5.1.12], 8. is a consequence of 2., 3., 5. and 6. For 9. see [Har77, PROPOSITION III.2.2]. For 10. see [Har66, THEOREM II.7.18].

II.1.2 Convention From now onwards, all schemes are assumed to be noetherian.

II.1.3 Notation For a scheme X we denote by $\mathbf{D}^*X$ the derived category of coherent sheaves $\mathbf{D}^*\mathsf{Coh}X$, where $* = \mathsf{ub}, +, -, \mathsf{b}$. Again for scheme X we adopt the following conventions:

$\mathbf{D}^*_{\mathsf{qc}}X = \mathbf{D}^*_{\mathsf{Qcoh}X}\mathcal{O}_X\text{-}\mathsf{Mod}$
(complexes of sheaves with quasi-coherent cohomology),
$\mathbf{D}^*_{\mathsf{c}}X = \mathbf{D}^*_{\mathsf{Coh}X}\mathsf{Qcoh}Y$
(complexes of quasi-coherent sheaves with coherent cohomology),

where as usual $* = \mathsf{ub}, +, -, \mathsf{b}$.

II.1.4 THEOREM
Let X *be a scheme.*

(II.1) The inclusion $\mathsf{Qcoh}X \hookrightarrow \mathcal{O}_X\text{-}\mathsf{Mod}$ *induces an equivalence*

$$\mathbf{D}^*\mathsf{Qcoh}X \xrightarrow{\sim} \mathbf{D}^*_{\mathsf{qc}}X$$

for $* = +, \mathsf{b}$.

(II.2) The inclusion $\mathsf{Coh}X \hookrightarrow \mathsf{Qcoh}X$ *induces an equivalence*

$$\mathbf{D}^{\mathsf{b}}X \xrightarrow{\sim} \mathbf{D}^{\mathsf{b}}_{\mathsf{c}}X.$$

Proof
As a particular case of THEOREM I.5.9 one has (II.1). For (II.2) see [Huy06, PROPOSITION 3.5].

II.2 A SELECTION OF DERIVED FUNCTORS

Following [Huy06, SECTION 3.3] we discuss the derived versions of the functors we employ in the sequel.

II.2.1 Convention From now onwards all schemes and morphisms are considered to be over a fixed field $\Bbbk$. Given two schemes X, Y we write $X \times Y$ for the product over $\Bbbk$: $X \times_{\Bbbk} Y$.

The general idea is to start with a given left (or right) exact functor

$$\mathrm{F} : \mathsf{Qcoh}\mathrm{X} \to \mathcal{B},$$

defined on quasi-coherent sheaves, and then to restrict the domain of its derived version $\mathbf{R}\mathrm{F}$ to the bounded derived category of coherent sheaves, finally one uses THEOREMS I.8.4 and II.1.4 to try and confine the codomain of $\mathbf{R}\mathrm{F}$ to a subcategory $\mathbf{D}^b_{\mathcal{C}}\mathcal{B}$, for some thick subcategory $\mathcal{C}$ of $\mathcal{B}$.

The first functor we treat is the *global sections* functor. Let X be a scheme. The global sections functor

Sheaf Cohomology

$$\begin{aligned} \Gamma(\mathrm{X}, -) = \Gamma : \mathsf{Qcoh}\mathrm{X} &\longrightarrow \Bbbk\text{-}\mathsf{Mod} \\ \mathscr{F} &\longmapsto \Gamma(\mathrm{X}, \mathscr{F}) = \mathscr{F}(\mathrm{X}) \end{aligned}$$

is left exact. Since Qcoh has enough injectives we obtain

$$\mathbf{R}\Gamma : \mathbf{D}^+\mathsf{Qcoh}\mathrm{X} \longrightarrow \mathbf{D}^+\Bbbk\text{-}\mathsf{Mod}.$$

For a sheaf $\mathscr{F}$, the module $\mathbf{R}^i\Gamma(\mathscr{F})$ is denoted by $\mathrm{H}^i(\mathrm{X}, \mathscr{F})$ and it is called the *i-th sheaf cohomology group* of $\mathscr{F}$. For an honest complex of sheaves $\mathscr{F}^\cdot$, the modules $\mathbf{R}^i(\mathrm{X}, \mathscr{F}^\cdot)$ are classically called *hypercohomology* groups, and again we denote them by $\mathrm{H}^i(\mathrm{X}, \mathscr{F}^\cdot)$. Since every complex of vector spaces splits we have a non-canonical isomorphism

$$\mathbf{R}\Gamma(\mathscr{F}^\cdot) \cong \bigoplus_i \mathrm{H}^i(\mathrm{X}, \mathscr{F}^\cdot)[-i]$$

in $\mathbf{D}(\Bbbk\text{-}\mathsf{Mod})$.

To restrict to the bounded subcategory we use the following theorem.

II.2.2 THEOREM (GROTHENDIECK)
Let $\mathscr{F}$ be a quasi-coherent sheaf on a noetherian scheme X. *Then*

$$\mathrm{H}^i(\mathrm{X}, \mathscr{F}) = 0$$

for $i >$ dim X.

Proof
See [Har77, THEOREM III.2.7]. ☙

Hence, by THEOREM I.8.4 the following is well defined:

$$\mathbf{R}\Gamma : \mathbf{D}^b\mathsf{Qcoh}\mathrm{X} \longrightarrow \mathbf{D}^b\Bbbk\text{-}\mathsf{Mod}.$$

The passage to the coherent realm has an additional property.

II.2.3 THEOREM
Let X *be a proper scheme over a field* $\Bbbk$ *and let* $\mathscr{F}$ *be a coherent sheaf on* X. *Then the cohomology groups* $\mathrm{H}^i(\mathrm{X}, \mathscr{F})$ *have all finite dimension over* $\Bbbk$.

Proof
It is a straightforward consequence of THEOREM II.2.5. ☙

Thus if X is proper over $\Bbbk$ we finally obtain

$$\mathbf{R}\Gamma : \mathbf{D}^b\mathrm{X} \to \mathbf{D}^b\Bbbk\text{-}\mathsf{Mod}^{\mathsf{f}}.$$

We summarize the above discussion with a diagram.

$$\begin{tikzcd}
\mathbf{D}^+\mathsf{Qcoh}X \arrow[r, "\mathbf{R}\Gamma"] & \mathbf{D}^+\Bbbk\text{-}\mathsf{Mod} \\
\mathbf{D}^b\mathsf{Qcoh}X \arrow[u, hook] \arrow[r, dashed] & \mathbf{D}^b\Bbbk\text{-}\mathsf{Mod} \arrow[u, hook] \\
\mathbf{D}^b\mathsf{Coh}X \arrow[u, hook] \arrow[r, dashed, "X \text{ proper}"] & \mathbf{D}^b\Bbbk\text{-}\mathsf{Mod}^f \arrow[u, hook]
\end{tikzcd}$$

Where, given a left noetherian ring R, we denote by R-Mod^f the thick abelian subcategory of R-Mod whose objects are finitely generated modules. We use an analogous notation for right modules.

Direct Image We now move on to pushforwards. Let $f : X \to Y$ be a morphism of schemes. The *direct image (or pushforward or pushdown)*

$$f_* : \mathsf{Qcoh}X \longrightarrow \mathsf{Qcoh}Y$$

is a left exact functor which yields

$$\mathbf{R}f_* : \mathbf{D}^+\mathsf{Qcoh}X \longrightarrow \mathbf{D}^+\mathsf{Qcoh}Y.$$

Given a complex of quasi-coherent sheaves $\mathscr{F}^\cdot$ on X, we define its *i-th higher direct image* as $\mathbf{R}^i f_*(\mathscr{F}^\cdot)$, which by definition is just

$$\mathscr{H}^i(\mathbf{R}f_*(\mathscr{F}^\cdot))$$

the i-th cohomology sheaf of the complex $\mathbf{R}f_*(\mathscr{F}^\cdot)$.

We now want to restrict to the bounded derived category.

II.2.4 Theorem

Let $f : X \to Y$ be a morphism of noetherian schemes and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Then the higher direct images $\mathbf{R}^i f_*\mathscr{F}$ are trivial for $i > \dim X$.

Proof

It is a consequence of [Har77, Proposition III.8.1] and of Theorem II.2.2. The first theorem states that the sheaf $\mathbf{R}^i f_*\mathscr{F}$ is none other than the sheafification of the presheaf

$$V \longmapsto H^i(f^{-1}V, \mathscr{F}|f^{-1}V),$$

and using the second theorem we conclude. ❧

Thus the restriction of the derived pushdown becomes

$$\mathbf{R}f_* : \mathbf{D}^b\mathsf{Qcoh}X \longrightarrow \mathbf{D}^b\mathsf{Qcoh}Y.$$

To deal with coherent sheaves we use the following fact.

II.2.5 Theorem

Let $f : X \to Y$ be a proper morphism of schemes, where Y is noetherian. Then, for any coherent sheaf $\mathscr{F}$, the higher direct images $\mathbf{R}^i f_*(\mathscr{F})$ are coherent. In particular $f_* : \mathsf{Coh}X \to \mathsf{Coh}Y$ is well-defined and left exact.

Proof
See [Gro67, Théorèm III.3.2.1 and Corollaire III.3.2.2]. ❧

In conclusion, for a proper morphism f, we have

$$\mathbf{R}f_* : \mathbf{D}^b\mathrm{X} \longrightarrow \mathbf{D}^b\mathrm{Y}.$$

We summarize the above with a diagram.

$$\begin{tikzcd}
\mathbf{D}^+\mathrm{QcohX} \arrow[r, "\mathbf{R}f_*"] & \mathbf{D}^+\mathrm{QcohY} \\
\mathbf{D}^b\mathrm{QcohX} \arrow[u, hook] \arrow[r, dashed] & \mathbf{D}^b\mathrm{QcohY} \arrow[u, hook] \\
\mathbf{D}^b\mathrm{CohX} \arrow[u, hook] \arrow[r, dashed, "f \text{ proper}"] & \mathbf{D}^b\mathrm{CohY} \arrow[u, hook]
\end{tikzcd}$$

II.2.6 Remark If $f : \mathrm{X} \to \operatorname{Spec}\Bbbk$ is the structural morphism, one notes that the direct image functor is particularly simple. The category of quasi-coherent sheaves over $\Bbbk$ is equivalent to the category of modules over $\Bbbk$. Through this equivalence the direct image functor corresponds to the global sections functor, in other words:

$$\mathbf{R}f_* = \mathbf{R}\Gamma \otimes_{\Bbbk} \mathcal{O}_k = \mathrm{H}^{\cdot}(\mathrm{X}, -) \otimes_{\Bbbk} \mathcal{O}_{\Bbbk}. \tag{II.3}$$

Given two morphisms of schemes $f : \mathrm{X} \to \mathrm{Y}$ and $g : \mathrm{Y} \to \mathrm{Z}$, we know that $(g \circ f)_* = g_* \circ f_*$. Therefore we have a morphism

$$\mathbf{R}(g \circ f)_* = \mathbf{R}\,(g_* \circ f_*) \longrightarrow \mathbf{R}g_* \circ \mathbf{R}f_* \tag{II.4}$$

to ensure that it is in fact an isomorphism we use flasque (a.k.a. flabby) sheaves, which form another class of f_*-injective objects in QcohX.

II.2.7 Lemma
On a ringed space, any injective $\mathcal{O}_{\mathrm{X}}$-module is flasque. Any flasque sheaf $\mathscr{F}$ on X is f_*-acyclic for any morphism $f : \mathrm{X} \to \mathrm{Y}$, i.e. $\mathbf{R}^i f_* \mathscr{F} = 0$ for $i > 0$. Moreover $f_*\mathscr{F}$ is again flasque.

Proof
See [Huy06, Lemma 3.24]. ❧

Indeed the morphism (II.4) is an isomorphism.

Deriving *inverse images (or pullbacks)* presents no difficulty to us as we only deal with flat morphisms. *Inverse Image*

II.2.8 Definition/Lemma
A morphism of schemes $f : \mathrm{X} \to \mathrm{Y}$ is flat *if for every $x \in \mathrm{X}$ the induced map $f_x^\sharp : \mathcal{O}_{\mathrm{Y},f(x)} \to \mathcal{O}_{\mathrm{X},x}$ is flat. If f is flat, then f^* is an exact functor.*

Proof
As a result of being left adjoint to the direct image functor f_*, the pullback f^* is in general right exact. Let f be a flat morphism and let $\mathscr{F} \to \mathscr{G}$ be an injective morphism of sheaves on Y. Consider the complex

$$0 \longrightarrow f^*\mathscr{F} \longrightarrow f^*\mathscr{G}$$

and its localization at any $x \in X$

$$0 \longrightarrow \mathscr{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \longrightarrow \mathscr{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \tag{II.5}$$

where the $\mathcal{O}_{Y,f(x)}$-module structure on $\mathcal{O}_{X,x}$ is precisely given from $f_x^\sharp$. Note that (II.5) is obtained by tensoring the exact sequence

$$0 \longrightarrow \mathscr{F}_{f(x)} \longrightarrow \mathscr{G}_{f(x)}$$

with the flat module $\mathcal{O}_{X,x}$, hence (II.5) is exact, which is enough to prove that f^* is exact.

Thanks to the above lemma we do not need to search for an f^*-projective class of objects, since f^* is automatically derivable and we denote $\mathbf{L}f^*$ simply by f^*.

II.2.9 Remark In the simple case where $f : X \to \operatorname{Spec} \Bbbk$ is the structural morphism the inverse image is quite simple. A quasi-coherent sheaf on $\Bbbk$ is of the form $V \otimes_\Bbbk \mathcal{O}_\Bbbk$, where V is a $\Bbbk$-vector space. The inverse image of such sheaf is simply

$$f^*(V \otimes_\Bbbk \mathcal{O}_\Bbbk) = V \otimes_\Bbbk \mathcal{O}_X. \tag{II.6}$$

Local Hom

We have two well-defined bifunctors

$$\mathscr{H}om_X : (\mathsf{Coh}X)^\circ \times \mathsf{Qcoh}X \longrightarrow \mathsf{Qcoh}X$$
$$\mathscr{H}om_X : (\mathsf{Coh}X)^\circ \times \mathsf{Coh}X \longrightarrow \mathsf{Coh}X$$

which in turn can be extended to complexes and pass on to the homotopic category:

$$\mathscr{H}om^\cdot_X : (\mathbf{K}^-\mathsf{Coh}X)^\circ \times \mathbf{K}^+\mathsf{Qcoh}X \longrightarrow \mathbf{K}^+\mathsf{Qcoh}X$$
$$\mathscr{H}om^\cdot_X : (\mathbf{K}^-\mathsf{Coh}X)^\circ \times \mathbf{K}^+\mathsf{Coh}X \longrightarrow \mathbf{K}^+\mathsf{Coh}X$$

where we use the $^\cdot$ to remind us that it is indeed a complex. After chasing around the identifications among opposite categories and complexes one notices that, given two complexes of sheaves $\mathscr{F}^\cdot$ and $\mathscr{G}^\cdot$, one has

$$\mathscr{H}om^n_X(\mathscr{F}^\cdot, \mathscr{G}^\cdot) = \bigoplus_i \mathscr{H}om_X(\mathscr{F}^i, \mathscr{G}^{i+n})$$

with differential given by

$$d^n|_{\mathscr{H}om(\mathscr{F}^i,\mathscr{G}^{i+n})}(\varphi) = \varphi \circ d^{i-1}_{\mathscr{F}^i} + (-1)^i d^{i+n}_{\mathscr{G}^{i+n}} \circ \varphi.$$

II.2.10 Remark Usually $\mathscr{H}om^n$ is defined as the complex

$$\mathscr{H}om^n(\mathscr{F}^\cdot, \mathscr{G}^\cdot) = \prod_i \mathscr{H}om(\mathscr{F}^i, \mathscr{G}^{i+n}).$$

However, in our cases the sums and products over i are actually finite and thus coincide.

Using the following lemma we may derive, obtaining

$$\mathbf{R}\mathscr{H}om^\cdot_X : (\mathbf{D}^-(\mathsf{Coh}X))^\circ \times \mathbf{D}^+\mathsf{Qcoh}X \to \mathbf{D}^+\mathsf{Qcoh}X$$

II.2.11 Lemma
Let $\mathscr{I}^\cdot$ be a bounded below complex of injective sheaves and let $\mathscr{F}^\cdot$ be another complex of sheaves. If $\mathscr{F}^\cdot$ or $\mathscr{I}^\cdot$ is acyclic then $\mathscr{H}om^\cdot(\mathscr{F}^\cdot, \mathscr{I}^\cdot)$ is acyclic.

Proof
See [Har66, Section II.3].

To restrict to bounded complexes we make some further assumptions on X. We assume X to be projective over $\Bbbk$ (and hence of finite dimension) and regular.

II.2.12 Lemma
If X is regular and projective, then any bounded complex of coherent sheaves is quasi-isomorphic to a bounded complex of locally free sheaves.

Proof
Since X is projective, every coherent sheaf is a quotient of a locally free sheaf (see [Har77, Corollary II.5.18]). We now want to use Lemma I.5.2, so we must prove that there exists a natural number d such that for any exact sequence of coherent sheaves

$$0 \to \mathscr{F} \to \mathscr{G}_1 \to \cdots \to \mathscr{G}_d$$

where the $\mathscr{G}_i$s are locally free, then $\mathscr{F}$ is also locally free. We fix $d \gg \dim X$. Since being locally free can be verified on stalks, we just need to prove that for any $x \in X$ the module $\mathscr{F}_x$ is locally free. We recall that for any $x \in X$, $\dim \mathcal{O}_{X,x} = \dim_p X \leq \dim X$. Thus we reduce to the case in which

$$0 \to F \to G_1 \to \cdots \to G_d$$

is an exact sequence of R-modules, where R is a noetherian regular local ring with $\dim R \leq \dim X$ and the G_is are free (and therefore projective) modules.

First we notice that every projective module over R is free (see [Eis99, Theorem A3.2]). Let M be $\operatorname{Coker}(G_{d-1} \to G_d)$ and let $P^\cdot$ be a projective resolution of F (which exists since R-Mod has enough projectives).

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & & & \\
 & P^{-1} \arrow[d] & & & & & \\
 & P^0 \arrow[d] \arrow[dr, dashed, "\varepsilon"] & & & & & \\
0 \arrow[r] & F \arrow[r] \arrow[d] & G_1 \arrow[r] & \cdots \arrow[r] & G_{d-1} \arrow[r] & M \arrow[r] & 0 \\
 & 0 & & & & &
\end{tikzcd}$$

Hence, through ε, we obtain a projective resolution of M, which starts with the G_is and with the P^{-j}s. We recall that the *global dimension* of a ring R is the supremum of the lengths of all projective resolutions of modules over R. We use the following facts:

- If R is regular local ring of dimension m, then the global dimension of R is m (see [Eis99, Corollary 19.6]).
- If R is a ring with global dimension m, then for any projective resolution $P^{\cdot}$ of any module M, the module $\mathrm{Im}(P^{-m} \to P^{-(m-1)})$ is projective (see [Eis99, Exercise A3.14]).

Thus there exists an integer $k \geq 1$ such that $\mathrm{Im}(G_k \to G_{k+1})$ is projective, and therefore free. Hence we have found an exact sequence

$$0 \to F \to Q_1 \to \cdots \to Q_r \to 0$$

where the Q_is are free. We claim that it now follows that F is also free, we use induction on r.

If $r = 1$ then F is isomorphic to Q_1 and hence it is free. Assume it is true for $r - 1$. Consider $N = \mathrm{Ker}(Q_{r-1} \to Q_r) = \mathrm{Im}(Q_{r-2} \to Q_{r-1})$. Since Q_r is projective, the sequence

$$0 \to N \to Q_{r-1} \to Q_r \to 0$$

splits, therefore N is a direct summand of a projective module, hence it is projective and thus free. As a consequence we have an exact sequence

$$0 \to F \to Q_1 \to \cdots \to Q_{r-2} \to N \to 0$$

and, by the induction hypothesis, we finally have that F is a free module over R.

We summarize with a diagram.

$$\begin{tikzcd}
(\mathbf{D}^{-}\mathrm{Coh}X)^{\circ} \times \mathbf{D}^{+}\mathrm{Qcoh}X \arrow[r, "\mathbf{R}\mathscr{H}om^{\cdot}"] & \mathbf{D}^{+}\mathrm{Qcoh}X \\
(\mathbf{D}^{-}\mathrm{Coh}X)^{\circ} \times \mathbf{D}^{+}\mathrm{Coh}X \arrow[u, hook] \arrow[r, dashed] & \mathbf{D}^{+}\mathrm{Coh}X \arrow[u, hook] \\
(\mathbf{D}^{b}\mathrm{Coh}X)^{\circ} \times \mathbf{D}^{b}\mathrm{Coh}X \arrow[u, hook] \arrow[r, dashed, "\text{X as above}"] & \mathbf{D}^{b}\mathrm{Coh}X \arrow[u, hook]
\end{tikzcd}$$

One defines

$$\mathscr{E}xt^i_X(\mathscr{F}^{\cdot}, \mathscr{G}^{\cdot}) = \mathbf{R}^i\mathscr{H}om^{\cdot}(\mathscr{F}^{\cdot}, \mathscr{G}^{\cdot}).$$

Dualization

As a special case one has the *derived dual* of a complex $\mathscr{F}^{\cdot} \in \mathbf{D}^{-}\mathrm{Qcoh}X$

$$(\mathscr{F}^{\cdot})^{\vee} = \mathbf{R}\mathscr{H}om(\mathscr{F}^{\cdot}, \mathcal{O}_X) \in \mathbf{D}^{+}\mathrm{Qcoh}X.$$

Of course the case that is of most interest to us is when $\mathscr{F}^{\cdot}$ is bounded and coherent: one has

$$\mathscr{F}^{\cdot\vee} \in \mathbf{D}^{b}X$$

if X is regular.

For *tensor products* we procede similarly as with local homs. One starts from the functor *Tensor Product*

$$- \otimes_{\mathcal{O}_X} - : \mathcal{O}_X\text{-Mod} \times \mathcal{O}_X\text{-Mod} \longrightarrow \mathcal{O}_X\text{-Mod}$$

which yields

$$\otimes_{\mathcal{O}_X} : \mathbf{K}^{-}\mathcal{O}_X\text{-Mod} \times \mathbf{K}^{-}\mathcal{O}_X\text{-Mod} \longrightarrow \mathbf{K}^{-}\mathcal{O}_X\text{-Mod},$$

where

$$(\mathscr{F}^{\cdot} \otimes_{\mathcal{O}_X} \mathscr{G}^{\cdot})^n = \bigoplus_{p+q=n} \mathscr{F}^p \otimes_{\mathcal{O}_X} \mathscr{G}^q$$

with differential

$$d_{|\mathscr{F}^p \otimes \mathscr{G}^q} = d^p_{\mathscr{F}^{\cdot}} \otimes 1 + (-1)^p 1 \otimes d^q_{\mathscr{G}^{\cdot}}.$$

To derive the tensor product we first use flat sheaves.

II.2.13 Lemma
Every $\mathcal{O}_X$-module is a quotient of a flat $\mathcal{O}_X$-module.

Proof
We recall that if $\mathscr{S}$ is a sheaf on an open subset U of X then its *extension by zero outside* U is the sheafification of the presheaf

$$V \longmapsto \begin{cases} \mathscr{S}(V) & \text{if } V \subset U, \\ 0 & \text{otherwise.} \end{cases}$$

We also recall the stalks of the extension of $\mathscr{S}$ are $\mathscr{S}_x$, on all $x \in U$, and vanish outside of U. Because of this last property, if we consider $\mathscr{F}_U$ to be the extension by zero of the restriction $\mathcal{O}_X|U$ of the structure sheaf on U, it is easy to check that $\mathscr{F}_U$ is a flat $\mathcal{O}_X$-module.

Again, we consider a sheaf $\mathscr{S}$ on X. Let $s \in \Gamma(U_s, \mathscr{S})$ be a section. Consider the sheaf homomorphism

$$\varphi_s : \mathscr{F}_{U_s} \to \mathscr{S}$$

induced by the presheaf morphism

$$(\varphi_s)_V(\lambda) = \lambda \cdot s|V \text{ if } V \subset U_s,$$
$$(\varphi_s)_V = 0 \text{ otherwise.}$$

We notice that for all $x \in U_s$, the germ s_x lies in the image of $(\varphi_s)_x$.

The morphisms φ_s thus induce a morphism

$$\mathscr{F} = \bigoplus_s \mathscr{F}_{U_s} \xrightarrow{\varphi} \mathscr{S}$$

where s rangese through all the sections of $\mathscr{S}$, which is clearly surjective on stalks. Since the direct sum of flat sheaves is flat we have constructed a surjective morphism from a flat sheaf onto $\mathscr{S}$. ❧

Since flat sheaves form (essentially by definition) the class $\otimes_{\mathcal{O}_X}$-projective sheaves we obtain

$$\otimes^{\mathbf{L}}_{\mathcal{O}_X} : \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \times \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \longrightarrow \mathbf{D}^-(\mathcal{O}_X\text{-Mod}).$$

Since the tensor product is right exact in both variables, we do not use injective objects, thus we move on directly to coherent sheaves. We remind that the tensor product of two coherent sheaves is again coherent. Again, we recall that if X is projective then CohX has enough locally frees. If we denote by $\mathcal{L}$ the full subcategory of CohX consisting of locally free sheaves, we have that the pair $(\mathcal{L}, \mathcal{L})$ is $\otimes$-projective. Hence we may derive on the left the tensor product by resolving by locally free sheaves.

If we assume moreover X to be regular we recall that any bounded complex of coherent sheaves is quasi-isomorphic to a bounded complex of locally free sheaves. As a consequence we can restrict to bounded complexes. We conclude with the usual diagram.

$$\begin{tikzcd}
\mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \times \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \arrow[r, "\otimes^{\mathbf{L}}_{\mathcal{O}_X}"] & \mathbf{D}^-(\mathcal{O}_X\text{-Mod}) \\
\mathbf{D}^-\mathrm{Coh}X \times \mathbf{D}^-\mathrm{Coh}X \arrow[r, dashed, "X \text{ projective}"] \arrow[u, hook] & \mathbf{D}^-\mathrm{Coh}X \arrow[u, hook] \\
\mathbf{D}^b\mathrm{Coh}X \times \mathbf{D}^b\mathrm{Coh}X \arrow[r, dashed, "X \text{ as above}"] \arrow[u, hook] & \mathbf{D}^b X \arrow[u, hook]
\end{tikzcd}$$

II.3 USEFUL ISOMORPHISMS

Tensor-Pullback Compatibility

There are some relationships among the derived functors above. Again we assume X to be regular and projective over $\Bbbk$. The simplest one we encounter is the compatibility between inverse images and tensor products.

II.3.1 LEMMA
Let $f : X \to Y$ be a morphism of ringed spaces. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on Y then there is an isomorphism

$$f^*\mathscr{F} \otimes f^*\mathscr{G} \simeq f^*(\mathscr{F} \otimes \mathscr{G}).$$

If $\mathscr{F}$ is locally free then $f^*\mathscr{F}$ is locally free.

Proof
The first statement is essentially due to the fact that the tensor product commutes with colimits in the category of modules (see [KS90, PROPOSITON 2.3.5]). The second assertion is a straightforward consequence of the following. By definition

$$f^*\mathcal{O}_Y = f^\bullet \mathcal{O}_Y \otimes_{f^\bullet \mathcal{O}_Y} \mathcal{O}_X \simeq \mathcal{O}_X$$

where $f^\bullet$ is the left adjoint of f_*, when considering sheaves of abelian groups and not just $\mathcal{O}_Y$-modules. ∾

To define $\otimes^{\mathbf{L}}$, on a projective scheme X, we used locally free sheaves. The above lemma thus yields the isomorphism:

$$f^*(\text{-}) \otimes^{\mathbf{L}} f^*(\text{-}) \xrightarrow{\sim} f^*(\text{-} \otimes^{\mathbf{L}} \text{-}). \tag{II.7}$$

The second one we study is the *projection formula*. *Projection Formula*

II.3.2 Lemma
Let $f : \mathrm{X} \to \mathrm{Y}$ be a morphism of ringed spaces and let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X *and* Y *respectively. If $\mathscr{G}$ is locally free then we have an isomorphism*

$$f_*(\mathscr{F}) \otimes \mathscr{G} \simeq f_*(\mathscr{F} \otimes f^*\mathscr{G}).$$

Proof
Using the unit and counit of the adjunction $f^* \dashv f_*$ we obtain the following chains of morphisms

$$\begin{aligned} f_*\mathscr{F} \otimes \mathscr{G} &\longrightarrow f_* f^*(f_*\mathscr{F} \otimes \mathscr{G}) \\ &\simeq f_*(f^* f_*\mathscr{F} \otimes f^*\mathscr{G}) \\ &\longrightarrow f_*(\mathscr{F} \otimes f^*\mathscr{G}). \end{aligned}$$

We now show that if $\mathscr{G}$ is locally free then the composition of the above maps is an isomrphism. Since the question is local, and we already have a globally defined map, we suppose $\mathscr{G} = \mathcal{O}_{\mathrm{Y}}^n$. The above map translates into

$$\begin{aligned} f_*\mathscr{F} \otimes \mathcal{O}_{\mathrm{Y}}^n \simeq (f_*\mathscr{F})^n &\longrightarrow (f_* f^* f_*\mathscr{F})^n \\ &\longrightarrow (f_*\mathscr{F})^n \simeq f_*\mathscr{F}^n \\ &\simeq f_*(\mathscr{F} \otimes \mathcal{O}_{\mathrm{X}}^n) \simeq f_*(\mathscr{F} \otimes f^*\mathcal{O}_{\mathrm{Y}}^n) \end{aligned}$$

which is an isomorphism (it is essentially equivalent to saying that f^* is left adjoint to f_*). ❧

Again we recall that to define $\otimes^{\mathbf{L}}$ (on a projective scheme) we used locally free sheaves, therefore the following isomorphism is a consequence of the above lemma:

$$\mathbf{R}f_*(\text{-}) \otimes^{\mathbf{L}} \text{-} \xrightarrow{\sim} \mathbf{R}f_*(\text{-} \otimes^{\mathbf{L}} f^*(\text{-})). \tag{II.8}$$

Let us begin by recalling a lemma. *Flat Base Change*

II.3.3 Lemma
Flatness and properness are stable under base change.

Proof
See [Liu02, Proposition 4.3.3] and [Liu02, Proposition 3.3.16]. ❧

Consider a cartesian diagram

$$\begin{tikzcd} \mathrm{X} \times_{\mathrm{Z}} \mathrm{Y} \arrow[r, "v"] \arrow[d, "g"'] & \mathrm{Y} \arrow[d, "f"] \\ \mathrm{X} \arrow[r, "u"] & \mathrm{Z} \end{tikzcd}$$

where f (and hence g) is proper. Using the inverse-direct image adjunction we obtain a functorial morphism

$$\begin{aligned} u^* f_* \xrightarrow{u^*\varepsilon} g_* g^* u^* f_* &= g_*(ug)^* f_* \\ &= g_*(fv)^* f_* \\ &= g_* v^* f^* f_* \xrightarrow{g_* v^* \delta} g_* v^* \end{aligned}$$

where ε is the unit relative to g and δ is the counit relative to f. Assume moreover u (and consequently v) to be flat, and let $\mathscr{F}$ be a quasi-coherent sheaf on Y. Then [Liu02, EXERCISE 1.16] yields the *flat base change* isomorphism

$$u^* f_* \longrightarrow g_* v^*$$

which passes on to the derived level

$$u^* \mathbf{R} f_* \xrightarrow{\sim} \mathbf{R} g_* v^*. \tag{II.9}$$

II.3.4 Remark A special case we will use later is the following. Consider the diagram

$$\begin{tikzcd} \mathrm{X} \times \mathrm{Y} \arrow[r, "p"] \arrow[d, "q"'] & \mathrm{X} \arrow[d, "f"] \\ \mathrm{Y} \arrow[r, "u"] & \Bbbk \end{tikzcd}$$

and an object $\mathscr{F}^\cdot \in \mathbf{D}^{\mathrm{b}}\mathrm{X}$. Combining (II.3), (II.6) and (II.9) one obtains

$$\begin{aligned} \mathbf{R} q_* p^* \mathscr{F}^\cdot = u^* \mathbf{R} f_* \mathscr{F}^\cdot &= u^*(\mathrm{H}^\cdot(\mathrm{X}, \mathscr{F}^\cdot) \otimes_{\Bbbk} \mathcal{O}_{\Bbbk}) \\ &= \mathrm{H}^\cdot(\mathrm{X}, \mathscr{F}^\cdot) \otimes_{\Bbbk} \mathcal{O}_{\mathrm{Y}}. \end{aligned} \tag{II.10}$$

II.4 FOURIER-MUKAI TRANSFORMS

In this section we introduce the concept of Fourier-Mukai transform. We do not delve deeply into the theory, we merely touch upon its definition and discuss the case needed in the next chapter.

Function Transforms

Let's start with an informal discussion (cf. [Tho99, SECTION 2]). Consider an integrable function $f \in \mathscr{L}^1(\mathbb{R}^n, \mathbb{C})$ defined on $\mathbb{R}^n$ with values in the complex numbers. The *Fourier transform of f*

$$\hat{f} : \mathbb{R}^n \to \mathbb{C}$$

is defined as

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) \cdot e^{-i\langle x, y\rangle} dx. \tag{II.11}$$

Let's write the exponential as $\mathcal{K}$ so that (II.11) becomes

$$\hat{f}(y) = \int_{\mathbb{R}^n} f(x) \cdot \mathcal{K}(x, y) dx. \tag{II.12}$$

Instead of the exponential we might take any other suitable function $\mathcal{K}$ in (II.12); for example $\mathcal{K} \in \mathcal{C}_c(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{C})$ (continuous with compact support). We can abstract further by considering, instead of two copies of $\mathbb{R}^n$, any pair of reasonable[1] measure spaces X and Y. Consequently, for suitable f and $\mathcal{K}$, we may define the *function transform with kernel* $\mathcal{K}$ of f as

$$\hat{f}(y) = \int_X f(x) \cdot \mathcal{K}(x, y) dx. \tag{II.13}$$

Since we wish to stress the dependence on the kernel we change notation and use $\Phi_{\mathcal{K}}(f)$ instead of $\hat{f}$.

The product X × Y comes equipped with two projections: p and q.

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

Let's introduce another two notations. Given a complex-valued function f on X we define the *pull-up* by p as

$$p^* f(x, y) = f \circ p(x, y) = f(x),$$

and given a complex-valued integrable function F on the product X × Y we define the *push-down* by q as

$$q_* F(y) = \int_X F(x, y) dx.$$

In other words the pull-up of a function f on X is simply a stack of copies of f on the product, constant along Y; the push-down of a function F on the product associates to every point $y \in Y$ the integral of F on the fiber $q^{-1}(y)$. With this last bit of notation in hand we may rewrite (II.13) as

$$\Phi_{\mathcal{K}}(f) = q_* \left(p^* f \cdot \mathcal{K} \right). \tag{II.14}$$

Of course we might as well have considered the pull-up by q and the push-down by p, which gives the opposite transform

$$\Phi'_{\mathcal{K}}(f) = p_* \left(q^* f \cdot \mathcal{K} \right).$$

Results concerning Fourier transforms may be expressed in this new language. For example the *inversion formula*; for an $\mathscr{L}^1$ function on $\mathbb{R}^n$ admitting an $\mathscr{L}^1$ Fourier transform let

$$g(x) = \left(\frac{1}{2\pi} \right)^n \int_{\mathbb{R}^n} \hat{f}(y) \cdot e^{i\langle x, y \rangle} dy$$

be the *antitransform* of f. Then the inversion formula states that $f = g$ almost everywhere, which immediately translates to

$$\Phi'_{\mathcal{H}} \circ \Phi_{\mathcal{K}}(f) = f \text{ almost everywhere,}$$

for appropriate kernels $\mathcal{K}$ and $\mathcal{H}$.

Fourier-Mukai Transforms

Coming back to sheaves, we formally transliterate the above definition of transform into the setting of derived categories of coherent sheaves. Let X and Y be two schemes, proper over $\Bbbk$, and consider their product $X \times Y$. We remark that since the structural morphisms from X and Y to $\operatorname{Spec} \Bbbk$ are flat, the projections p and q are also flat.

II.4.1 Definition
Given an element $\mathscr{K} \in \mathbf{D}^b(X \times Y)$ *we define the* Fourier-Mukai transform with kernel $\mathscr{K}$ *to be the functor*

$$\begin{aligned} \Phi_{\mathscr{K}} : \mathbf{D}^b X &\longrightarrow \mathbf{D}^b Y \\ \mathscr{F}^{\cdot} &\longmapsto \mathbf{R}q_* \left(\mathbf{L}p^* \mathscr{F}^{\cdot} \otimes^{\mathbf{L}} \mathscr{K} \right). \end{aligned}$$

A functor as such is sometimes called an *integral transform*, gaining the full title of *Fourier-Mukai* only when it is an equivalence.

II.4.2 Remark Since we only deal with flat morphisms the definition of Fourier-Mukai transform becomes

$$\mathbf{R}q_* \left(p^* \mathscr{F}^{\cdot} \otimes^{\mathbf{L}} \mathscr{K} \right).$$

Let us give at once an example. Consider a morphism $f : X \to Y$. The maps Id_X and f uniquely determine a map

$$\iota = \mathrm{Id}_X \times f : X \to X \times Y.$$

Let $\mathcal{O}_{\Gamma_f} = \iota_* \mathcal{O}_X$ be the direct image of the structure sheaf of X through ι (viz. it is the structure sheaf of the graph Γ_f of f). Consider now the Fourier-Mukai transform with kernel $\mathcal{O}_{\Gamma_f}$:

$$\begin{aligned} \Phi_{\mathcal{O}_{\Gamma_f}} &= \mathbf{R}q_* (p^* - \otimes^{\mathbf{L}} \mathcal{O}_{\Gamma_f}) \\ &= \mathbf{R}q_* (p^* - \otimes^{\mathbf{L}} \iota_* \mathcal{O}_X) \\ \text{(projection formula)} &= \mathbf{R}q_* \mathbf{R}\iota_* (\iota^* p^* - \otimes^{\mathbf{L}} \mathcal{O}_X) \\ (p\iota = \mathrm{Id}_X,\ q\iota = f) &= \mathbf{R}f_* \end{aligned}$$

and the opposite transform

$$\begin{aligned} \Phi'_{\mathcal{O}_{\Gamma_f}} &= \mathbf{R}p_* (q^* - \otimes^{\mathbf{L}} \mathcal{O}_{\Gamma_f}) \\ \text{(projection formula)} &= \mathbf{R}p_* \mathbf{R}\iota_* (\iota^* q^* - \otimes^{\mathbf{L}} \mathcal{O}_X) \\ (p\iota = \mathrm{Id}_X,\ q\iota = f) &= f^*. \end{aligned}$$

Thus they are simply the direct image and the inverse image through f. As a special case we notice the following. Let $X = Y$ and $f = \mathrm{Id}_X$, so that $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal, then

$$\Phi_{\mathcal{O}_\Delta} = \mathrm{Id}_{\mathbf{D}^b X} = \Phi'_{\mathcal{O}_\Delta}. \tag{II.15}$$

To conclude we note that a Fourier-Mukai transform, being the composition of Δ-functors, is itself a Δ-functor.

1 For example we request the spaces to be σ-finite, in order to be able to apply the Fubini-Tonelli theorem (see [Rud87, Theorem 8.8]).

III

BEĬLINSON'S THEOREM

Contents

In this chapter we finally come to investigate the structure of Coh$\mathbb{P}$, the category of coherent sheaves on projective space. More precisely we study $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}\mathbb{P})$, the bounded derived category of Coh$\mathbb{P}$, by means of a resolution of the structure sheaf of the diagonal. Applying the machinery of Fourier-Mukai transforms we obtain two equivalences of $\mathbf{D}^{\mathrm{b}}\mathbb{P}$ with simpler, more algebraic, triangulated categories.

Notation

Let $\mathbb{P} = \mathbb{P}(\mathrm{V}) = \mathrm{Proj}\,\mathrm{Sym}\,\mathrm{V}^\vee$ be the projective space associated with an $(n+1)$-dimensional vector space V over a field $\mathbb{k}$. We write $\mathbf{D}$ for $\mathbf{D}^{\mathrm{b}}(\mathrm{Coh}\mathbb{P})$. Let p and q be the projections

$$\begin{tikzcd} & \mathbb{P} \times \mathbb{P} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P} & & \mathbb{P} \end{tikzcd}$$

from the product $\mathbb{P} \times \mathbb{P}$ onto the first and second factor respectively. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}}$ denote the structure sheaf of $\mathbb{P}$, let $\Omega^1 = \Omega^1_{\mathbb{P}/\mathbb{k}}$ be the sheaf of differentials over $\mathbb{k}$ and let $\mathcal{O}_\Delta$ be the direct image of $\mathcal{O}$ through the diagonal map (i.e. $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset \mathbb{P} \times \mathbb{P}$). If $\mathscr{F}$ and $\mathscr{G}$ are sheaves on $\mathbb{P}$, we define the *exterior tensor product* of $\mathscr{F}$ and $\mathscr{G}$ to be the sheaf on the product $\mathbb{P} \times \mathbb{P}$

$$\mathscr{F} \boxtimes \mathscr{G} = p^*\mathscr{F} \otimes q^*\mathscr{G}.$$

Projective Space

To ensure that we can employ the machinery developed in the last chapter we make a few remarks (see [Liu02]). The schemes $\mathbb{P}$ and $\mathbb{P} \times \mathbb{P}$ are notherian, of finite dimension, projective over $\mathbb{k}$, smooth and hence regular. The structural morphism $\mathbb{P} \to \mathrm{Spec}\,\mathbb{k}$ is proper and flat, whence it follows that the projections

$$p, q : \mathbb{P} \times \mathbb{P} \longrightarrow \mathbb{P}$$

are also proper and flat. Thus we can apply all the results of the previous chapter.

III.1 A RESOLUTION FOR THE DIAGONAL

In this section we will construct a finite locally free resolution of the structure sheaf of the diagonal $\mathcal{O}_\Delta$. It is a special feature of projective space that such a resolution exists. For the machinery of Koszul complexes we refer to [FL85, IV§-2].

III.1.1 Theorem (Beĭlinson)
Consider the locally free sheaf

$$\mathscr{S} = \mathcal{O}(-1) \boxtimes \Omega^1(1)$$

on $\mathbb{P} \times \mathbb{P}$. There is a finite locally free resolution of the sheaf $\mathcal{O}_\Delta$ on $\mathbb{P} \times \mathbb{P}$, given by

$$0 \to \bigwedge^n \mathscr{S} \to \cdots \to \bigwedge^2 \mathscr{S} \to \mathscr{S} \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_\Delta \to 0$$

Proof
We sketch a first proof and then give second proof.

First Proof

We recall that $\mathcal{O}(-1)$ is the tautological bundle, whose fiber at a point $l \in \mathbb{P}$ is l itself, considered as a subspace of V. The sheaf $\Omega^1(1)$ is the dual of $\mathscr{T}(-1)$, the tangent sheaf twisted by -1. Its fiber at l consists of the linear maps from V to $\Bbbk$ vanishing on l. By pulling these two sheaves back, via the two projections p and q, we obtain the sheaf $\mathscr{S}$, whose fiber at $(l_1, l_2) \in \mathbb{P} \times \mathbb{P}$ is the tensor product of l_1 with the space of linear maps from V to $\Bbbk$ vanishing on l_2.

We may therefore construct the *evaluation* morphism

$$\varepsilon : \mathscr{S} \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}}$$

defined as (the $\Bbbk$-linear extension of) the evaluation map

$$\varepsilon(v \otimes \varphi) = \varphi(v)$$

where v is an element of l_1 and φ vanishes on l_2.

One observes at once that $\varepsilon_{(l_1,l_2)}$ ceases to be surjective if and only if $l_1 = l_2$. The image of ε and the ideal sheaf of the diagonal $\Delta \subset \mathbb{P} \times \mathbb{P}$ cut out the same subspace. It remains to prove that they indeed determine the same scheme. To do that, one works locally (se second proof below). Therefore $\mathcal{O}_\Delta$ is the cokernel of ε, and we have the exact sequence

$$\mathscr{S} \xrightarrow{\ \varepsilon\ } \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

which in turn yields the Koszul complex

$$0 \to \bigwedge^n \mathscr{S} \to \cdots \to \bigwedge^2 \mathscr{S} \to \mathscr{S} \xrightarrow{\varepsilon} \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_\Delta \to 0$$

where the map

$$\bigwedge^p \mathscr{S} \longrightarrow \bigwedge^{p-1} \mathscr{S}$$

is given by

$$s_1 \wedge \cdots \wedge s_k \longmapsto \sum_{j=1}^{p} (-1)^{j-1} \varepsilon(s_p) s_1 \wedge \cdots \wedge \widehat{s_j} \wedge \cdots \wedge s_k.$$

Since the diagonal has codimension n in $\mathbb{P} \times \mathbb{P}$ the above complex is exact. We may therefore speak of a *Koszul resolution*.

Second Proof

Alternatively, we may proceed as follows. We fix an isomorphism of V with $\mathbb{k}^{n+1}$, thus $\mathbb{P}$ becomes the Proj of a polynomial ring. We consider three copies of $\mathbb{P}$ all labeled with different variables:

$$\begin{aligned} \mathbb{P}_x &= \operatorname{Proj} \mathbb{k}[x_0, \dots, x_n] \\ \mathbb{P}_y &= \operatorname{Proj} \mathbb{k}[y_0, \dots, y_n] \\ \mathbb{P}_z &= \operatorname{Proj} \mathbb{k}[z_0, \dots, z_n] \end{aligned}$$

and we consider the product $\mathbb{P}_x \times \mathbb{P}_y$ with projections

$$\begin{tikzcd} & \mathbb{P}_x \times \mathbb{P}_y \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathbb{P}_x & & \mathbb{P}_y \end{tikzcd}$$

Following the proof of [Har77, THEOREM II.8.13] let's consider the Euler sequence

$$0 \longrightarrow \Omega_{\mathbb{P}}(1) \longrightarrow \mathcal{O}_{\mathbb{P}}^{n+1} \longrightarrow \mathcal{O}_{\mathbb{P}}(1) \longrightarrow 0.$$

We want to take the first morphism on $\mathbb{P}_x$ and the second morphism on $\mathbb{P}_y$, pull them back through q and p respectively to obtain the map below

$$q^* \Omega_{\mathbb{P}_x}(1) \to q^* \mathcal{O}_{\mathbb{P}_x}^{n+1} \simeq \mathcal{O}_{\mathbb{P}\times\mathbb{P}}^{n+1} \simeq p^* \mathcal{O}_{\mathbb{P}_y}^{n+1} \to p^* \mathcal{O}_{\mathbb{P}_y}(1) \tag{III.1}$$

which in turn can be tensored by $p^* \mathcal{O}_{\mathbb{P}_y}(-1)$ yielding a homomorphism

$$p^* \mathcal{O}_{\mathbb{P}_y}(-1) \otimes_{\mathcal{O}_{\mathbb{P}\times\mathbb{P}}} q^* \Omega_{\mathbb{P}_x}(1) \longrightarrow \mathcal{O}_{\mathbb{P}\times\mathbb{P}}. \tag{III.2}$$

We claim that the image of (III.2) is the ideal sheaf determining the diagonal. To prove our claim we work on local coordinate patches.

Denote by e_i the standard section $(0, \dots, 0, 1, 0, \dots, 0)$ of $\mathcal{O}^{n+1}$. On $\mathrm{U} = \{x_i \neq 0\}$ we have a basis of $\Gamma(\mathrm{U}, \Omega_{\mathbb{P}_x}(1))$ such that under the first morphism in the Euler sequence the k-th basis element maps to

$$\left(e_k - \frac{x_k}{x_i} e_i \right)$$

for $k \neq i$. On $\mathbb{P}_y$ the second morphism of the Euler sequence is precisely given by

$$e_k \longmapsto y_k.$$

Thus, if we work on the coordinate patch $U \times V = \{x_i \neq 0, y_j \neq 0\}$, the morphism (III.1) maps the k-th basis element of the global sections of $q^*\Omega_{\mathbb{P}_x}(1)$ to

$$y_k - \frac{x_k}{x_i} y_i = y_j \left(\frac{y_k}{y_j} - \frac{y_i}{y_j} \frac{x_k}{x_i} \right).$$

Tensoring by $p^*\mathcal{O}_{\mathbb{P}_y}(-1)$, as in (III.2), we again obtain a basis of the sections over $U \times V$ of the first sheaf such that the k-th basis element maps to

$$\frac{y_k}{y_j} - \frac{y_i}{y_j} \frac{x_k}{x_i} \tag{III.3}$$

for $k \neq i$.

To prove our claim, that the image of (III.2) is in fact the ideal sheaf of the diagonal, we consider the third projective space $\mathbb{P}_z$ and the diagonal map $\mathbb{P}_z \to \mathbb{P}_y \times \mathbb{P}_x$. Since we restricted our attention to $U \times V$, we consider $U \cap V$ in $\mathbb{P}_z$ which is simply

$$\operatorname{Spec} \Bbbk \left[\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}, \frac{z_0}{z_j}, \dots, \frac{z_n}{z_j} \right] = \operatorname{Spec} \Bbbk \left[\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}, \frac{z_i}{z_j} \right].$$

The diagonal map thus corresponds to

$$\begin{aligned} \Bbbk \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \dots, \frac{y_n}{y_j} \right] &\longrightarrow \Bbbk \left[\frac{z_0}{z_i}, \dots, \frac{z_n}{z_i}, \frac{z_i}{z_j} \right] \\ \frac{x_k}{x_i} &\longmapsto \frac{z_k}{z_i} \\ \frac{y_k}{y_j} &\longmapsto \frac{z_k}{z_j}. \end{aligned}$$

We now show that the kernel of the above surjective morphism is precisely the ideal generated by the elements of the form (III.3), hence proving our claim. Call $\mathcal{R}$ the ideal generated by the elements of the form (III.3). Since $\mathcal{R}$ maps to zero (by inspection) the above morphism passes onto a morphism φ from the quotient ring

$$\mathcal{Q} = \frac{\Bbbk \left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{y_0}{y_j}, \dots, \frac{y_n}{y_j} \right]}{\mathcal{R}}.$$

For $k \neq i$ we have the identies

$$\frac{y_k}{y_j} = \frac{y_i}{y_j} \frac{x_k}{x_i}$$

holding in $\mathcal{Q}$. In the case $k = j$ this yields

$$1 = \frac{y_j}{y_j} = \frac{y_i}{y_j} \frac{x_j}{x_i} \tag{III.4}$$

thus

$$\frac{y_i}{y_j} = \left(\frac{x_j}{x_i} \right)^{-1}$$

in $\mathcal{Q}$, and so

$$\mathcal{Q} \cong \Bbbk\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}, \frac{x_i}{x_j}\right]$$

from which it follows that φ is an isomorphism. Hence our claim.

Taking the Koszul resolution associated with the morphism (III.2) we conlude. ❧

Let's write the resolution again. We have a canonical identification

$$\bigwedge^k \mathscr{S} = \bigwedge^k \big(\mathcal{O}(-1) \boxtimes \Omega^1(1)\big) \simeq \mathcal{O}(-k) \boxtimes \Omega^k(k)$$

where the last isomorphism is a consequence of the following simple algebraic lemma.

III.1.2 Lemma

Let R *be a ring, let* M *and* N *be free* R*-modules of finite rank. Assume moreover* M *to have rank one. Then we have a natural isomorphism*

$$\bigwedge^r (\mathrm{M} \otimes \mathrm{N}) \simeq \mathrm{M}^{\otimes r} \otimes \bigwedge^r \mathrm{N}.$$

Proof

We define a map (as the $\Bbbk$-linear extension of)

$$\begin{aligned} \mathrm{M}^{\otimes r} \otimes \bigwedge^r \mathrm{N} \ni (m_1 \otimes \cdots \otimes m_r) \otimes (n_1 \wedge \cdots \wedge n_r) \\ \longmapsto (m_1 \otimes n_1) \wedge \cdots \wedge (m_r \otimes n_r) \in \bigwedge^r (\mathrm{M} \otimes \mathrm{N}) \end{aligned}$$

which is well-defined since M is of rank one. Clearly it is surjective and therefore, being M and N free, an isomorphism. ❧

As a result, if we write $\mathscr{L}^{-k}$ for $\mathcal{O}(-k) \boxtimes \Omega^k(k)$, the Koszul resolution above becomes

$$0 \to \mathscr{L}^{-n} \to \mathscr{L}^{-n+1} \to \cdots \to \mathscr{L}^{-2} \to \mathscr{L}^{-1} \to \mathscr{L}^{0} \to \mathcal{O}_\Delta \to 0$$

so the complex $\mathscr{L}^{\cdot}$ is quasi-isomorphic to the 0-complex $\mathcal{O}_\Delta$. Thus the two are isomorphic in the derived category $\mathbf{D}^{\mathrm{b}}(\mathbb{P} \times \mathbb{P})$.

III.2 FOURIER-MUKAI KERNELS

By writing $\Phi_{\mathcal{K}}$ we denote the Fourier-Mukai transform with kernel $\mathcal{K}$, going from the first factor to the second; by $\Phi'_{\mathcal{K}}$ we indicate the same transform, but going in the opposite direction.

Let $\mathscr{F}^{\cdot}$ be a bounded complex of coherent sheaves on $\mathbb{P}$. The quasi-isomorphism $\mathscr{L}^{\cdot} \simeq \mathcal{O}_\Delta$ thus gives

$$\mathscr{F}^{\cdot} \simeq \Phi_{\mathcal{O}_\Delta}(\mathscr{F}^{\cdot}) \simeq \Phi_{\mathscr{L}^{\cdot}}(\mathscr{F}^{\cdot})$$

and

$$\mathscr{F}^{\cdot} \simeq \Phi'_{\mathcal{O}_\Delta}(\mathscr{F}^{\cdot}) \simeq \Phi'_{\mathscr{L}^{\cdot}}(\mathscr{F}^{\cdot})$$

isomorphisms which of course occur in $\mathbf{D}$. One may view them as a *decomposition* of the identity, since $\mathscr{L}^{\cdot}$ is an honest *complex* of sheaves, and not simply a sheaf (cf. THEOREM III.2.1). So each sheaf $\mathscr{F}$ can be *decomposed* in terms of the $\mathcal{O}(-r) \boxtimes \Omega^r(r)$s (the cochains of $\mathscr{L}^{\cdot}$).

Let us analyze each Fourier-Mukai transfrom $\Phi_{\mathscr{L}^{-r}}$. We have

$$\begin{aligned}
\Phi_{\mathscr{L}^{-r}}(\mathscr{F}^{\cdot}) &= \mathbf{R}q_*\left(\mathbf{L}p^*\mathscr{F}^{\cdot} \otimes^{\mathbf{L}} \mathscr{L}^{-r}\right) \\
(\text{flatness of } p) &\simeq \mathbf{R}q_*\left(p^*\mathscr{F}^{\cdot} \otimes^{\mathbf{L}} \left(p^*\mathcal{O}(-r) \otimes q^*\Omega^r(r)\right)\right) \\
(\text{locally freeness}) &\simeq \mathbf{R}q_*\left(p^*\mathscr{F}^{\cdot}(-r) \otimes q^*\Omega^r(r)\right) \\
(\text{projection formula}) &\simeq \mathbf{R}q_* p^*\mathscr{F}^{\cdot}(-r) \otimes \Omega^r(r) \\
(\text{flat base change}) &\simeq \mathbf{R}\Gamma\left(\mathbb{P}, \mathscr{F}^{\cdot}(-r)\right) \otimes_{\mathbb{k}} \mathcal{O} \otimes \Omega^r(r) \\
&\simeq \mathrm{H}^{\cdot}\left(\mathbb{P}, \mathscr{F}^{\cdot}(-r)\right) \otimes_{\mathbb{k}} \Omega^r(r).
\end{aligned} \tag{III.5}$$

Similarly

$$\Phi'_{\mathscr{L}^{-r}}(\mathscr{F}^{\cdot}) \simeq \mathrm{H}^{\cdot}\left(\mathbb{P}, \mathscr{F}^{\cdot} \otimes \Omega^r(r)\right) \otimes_{\mathbb{k}} \mathcal{O}(-r). \tag{III.6}$$

There is a striking similarity between (III.5) and (III.6) above and the relationship between a basis of a vector space and its dual basis. Let's elaborate some more on this last point. Given a basis of a finite dimensional vector space

$$v_1, \ldots, v_d$$

we know that there is a *dual basis*

$$v_1^{\vee}, \ldots, v_d^{\vee}$$

on the dual vector space, defined by the equations

$$v_i^{\vee}(v_j) = \delta_{ij}.$$

Any vector v can be expressed uniquely as a sum

$$v = \sum_{j=1}^{d} \lambda_j v_j$$

and it is straightforward to check that λ_j is actually $v_j^{\vee}(v)$, in other words

$$v = \sum_{j=1}^{d} v_j^{\vee}(v) v_j. \tag{III.7}$$

The Fourier-Mukai transforms above express a sort of *duality* between the sheaves

$$\mathcal{O}, \ldots, \mathcal{O}(n)$$

and the sheaves

$$\mathcal{O}, \Omega^1(1), \ldots, \Omega^n(n).$$

One would hope to obtain identities similar to (III.7) for coherent sheaves on $\mathbb{P}$, as:

$$\mathscr{F} \text{`}\simeq\text{'} \bigoplus_{j=0}^{n} \mathrm{H}^{\cdot}(\mathbb{P}, \mathscr{F}(-j)) \otimes \Omega^j(j)[j];$$

$$\mathscr{F} \text{`}\simeq\text{'} \bigoplus_{j=0}^{n} \mathrm{H}^{\cdot}(\mathbb{P}, \mathscr{F} \otimes \Omega^j(j)) \otimes \mathcal{O}(-j)[j].$$

Unfortunately the case is not so simple (but neither so dissimilar): separating the left and right hand sides above lie two spectral sequences[1].

III.2.1 Theorem
Given a coherent sheaf $\mathscr{F}$ on $\mathbb{P}$ there exist two spectral sequences:

$$\mathrm{E}_1^{r,s} = \mathrm{H}^s(\mathbb{P}, \mathscr{F}(r)) \otimes \Omega^{-r}(-r) \implies \begin{cases} \mathscr{F} & \text{if } r+s=0; \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{E}_1^{r,s} = \mathrm{H}^s(\mathbb{P}, \mathscr{F} \otimes \Omega^{-r}(-r)) \otimes \mathcal{O}(r) \implies \begin{cases} \mathscr{F} & \text{if } r+s=0; \\ 0 & \text{otherwise.} \end{cases}$$

***Proof** (cf. [Huy06, Proposition 8.28])*
Both are a consequence of the spectral sequence

$$\mathrm{E}_1^{r,s} = \mathbf{R}^s\mathrm{F}(\mathrm{A}^r) \implies \mathbf{R}^{r+s}\mathrm{F}(\mathrm{A}^\cdot)$$

for any bounded complex $\mathrm{A}^\cdot$. We only show the first case.
Let $\mathrm{A}^\cdot$ be $p^*(\mathscr{F}) \otimes \mathscr{L}^\cdot$ and let F be q_*. Therefore, using (III.5),

$$\mathbf{R}^s\mathrm{F}(\mathrm{A}^r) \simeq \mathrm{H}^s(\mathbb{P}, \mathscr{F}(r)) \otimes \Omega^{-r}(-r).$$

Finally

$$\begin{aligned} \mathbf{R}^{r+s}\mathrm{F}(\mathrm{A}) &= \mathscr{H}^{r+s}(\Phi_{\mathscr{L}^\cdot}(\mathscr{F})) \\ &= \mathscr{H}^{r+s}(\Phi_{\mathcal{O}_\Delta}(\mathscr{F})) \\ &\simeq \mathscr{H}^{r+s}(\mathscr{F}) \\ &= \begin{cases} \mathscr{F} & \text{if } r+s=0; \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

which concludes the proof.

III.3 TRIANGULATED MACHINERY

III.3.1 Definition
Let K be a $\Bbbk$-linear triangulated category.

- *An object $\mathrm{E} \in \mathsf{K}$ is* exceptional *if*

$$\mathrm{Hom}_{\mathsf{K}}(\mathrm{E}, \mathrm{E}[l]) = \begin{cases} \Bbbk & \text{if } l = 0; \\ 0 & \text{otherwise.} \end{cases}$$

- *A sequence of objects*

$$\mathrm{E}_1, \ldots, \mathrm{E}_n$$

is exceptional *if*

$$\mathrm{Hom}_{\mathsf{K}}(\mathrm{E}_i, \mathrm{E}_j[l]) = \begin{cases} \Bbbk & \text{if } l = 0, i = j; \\ 0 & \text{if } l \neq 0, i = j; \\ 0 & \text{if } i > j. \end{cases}$$

Note that every E_i is exceptional.

1 We also remark that using *Postnikov systems* one may view the sheaf $\mathscr{F}$ on the left as an *iterated cone* of the summands on the right. The idea is precisely the one employed in the proof of Theorem III.3.2, when splitting the Koszul resolution into short exact sequences.

- *A sequence is* full *if it generates* K *(i.e. any full triangulated subcategory containing the sequence is equivalent, via the inclusion, to* K*).*
- *A sequence is* strong *if*

$$\mathrm{Hom}_{\mathsf{K}}(\mathrm{E}_i, \mathrm{E}_j[l]) = \begin{cases} \Bbbk & \text{if } l = 0, i = j; \\ 0 & \text{if } l \neq 0. \end{cases}$$

The following theorem is of great importance and is fundamental for the sections below.

III.3.2 Theorem
The sheaves

$$\mathcal{O}(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}$$

form a strong full exceptional sequence in $\mathbf{D}^{\mathrm{b}}\mathbb{P}$.

Proof (cf. [Huy06, Corollary 8.29])
Denote by E_j the j-th term of the sequence $\{\mathcal{O}(-n), \mathcal{O}(-n+1), \ldots, \mathcal{O}\}$, thus

$$\mathrm{E}_j = \mathcal{O}(j-n-1).$$

Let's start by showing that the sequence is strong and exceptional.

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}}(\mathrm{E}_i, \mathrm{E}_j[l]) &= \mathrm{Hom}_{\mathbf{D}}\left(\mathcal{O}(i-n-1), \mathcal{O}(j-n-1)[l]\right) \\ &= \mathrm{Ext}^l_{\mathcal{O}}\left(\mathcal{O}(i-n-1), \mathcal{O}(j-n-1)\right) \\ &\simeq \mathbf{R}^l\mathrm{Hom}_{\mathcal{O}}\left(\mathcal{O}(i-n-1), \mathcal{O}(j-n-1)\right) \\ &\simeq \mathbf{R}^l\Gamma\left(\mathbb{P}, \mathcal{O}(j-i)\right) \\ &\simeq \mathrm{H}^l(\mathbb{P}, \mathcal{O}(j-i)) \\ &\simeq \begin{cases} 0 & \text{if } l \neq 0; \\ \mathrm{Sym}^{j-i}\mathrm{V}^\vee & \text{if } l = 0 \end{cases} \end{aligned}$$

which in particular shows that

$$\mathrm{Hom}_{\mathbf{D}}(\mathrm{E}_i, \mathrm{E}_j[l]) \simeq \begin{cases} \Bbbk & \text{if } l = 0 \;\&\; i = j; \\ 0 & \text{if } l = 0 \;\&\; i > j \end{cases}$$

therefore the sequence is strong and exceptional. Let us now show that it is indeed full. Let's write the Koszul resolution of the diagonal again.

$$0 \to \mathscr{L}^{-n} \to \cdots \to \mathscr{L}^{-1} \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \to \mathcal{O}_\Delta \to 0$$

We can split it up into short exact sequences

$$0 \longrightarrow \mathscr{L}^{-n} \longrightarrow \mathscr{L}^{-n+1} \longrightarrow \mathscr{M}_{-n+1} \longrightarrow 0$$

$$0 \longrightarrow \mathscr{M}_{-n+1} \longrightarrow \mathscr{L}^{-n+2} \longrightarrow \mathscr{M}_{-n+2} \longrightarrow 0$$

$$\vdots$$

$$0 \longrightarrow \mathscr{M}_{-1} \longrightarrow \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

which can be regarded as distinguished triangles in $\mathbf{D}$. Let now $\mathscr{F}^{\cdot} \in \mathbf{D}$ be an element of the bounded derived category of $\mathsf{Coh}\mathbb{P}$. We notice that $\mathbf{R}p_*$ and $-\otimes^{\mathbf{L}} q^*\mathscr{F}^{\cdot}$, being exact functors, send distinguished triangles into distinguished triangles. By applying $-\otimes^{\mathbf{L}} q^*\mathscr{F}^{\cdot}$ first and $\mathbf{R}p_*$ second we obtain distinguished triangles

$$
\begin{tikzcd}
\Phi'_{\mathscr{L}_{-n}}(\mathscr{F}^{\cdot}) \arrow[r] & \Phi'_{\mathscr{L}_{-n+1}}(\mathscr{F}^{\cdot}) \arrow[r] & \Phi'_{\mathscr{M}_{-n+1}}(\mathscr{F}^{\cdot}) \arrow[r, "+1"] & {} \\
\Phi'_{\mathscr{M}_{-n+1}}(\mathscr{F}^{\cdot}) \arrow[r] & \Phi'_{\mathscr{L}_{-n+2}}(\mathscr{F}^{\cdot}) \arrow[r] & \Phi'_{\mathscr{M}_{-n+2}}(\mathscr{F}^{\cdot}) \arrow[r, "+1"] & {} \\
 & \vdots & & \\
\Phi'_{\mathscr{M}_{-1}}(\mathscr{F}^{\cdot}) \arrow[r] & \Phi'_{\mathcal{O}_{\mathbb{P}\times\mathbb{P}}}(\mathscr{F}^{\cdot}) \arrow[r] & \Phi'_{\mathcal{O}_{\Delta}}(\mathscr{F}^{\cdot}) \arrow[r, "+1"] & {}
\end{tikzcd}
$$

of which the r-th row is

$$
\begin{tikzcd}
 & \Phi'_{\mathscr{M}_{-n+r}}(\mathscr{F}^{\cdot}) \arrow[dl, "+1"'] & \\
\Phi'_{\mathscr{M}_{-n+r-1}}(\mathscr{F}^{\cdot}) \arrow[rr] & & \mathrm{H}^{\cdot}(\mathbb{P}, \mathscr{F}^{\cdot} \otimes \Omega^{n-r}(n-r)) \otimes_{\Bbbk} \mathrm{E}_{r+1} \arrow[ul]
\end{tikzcd}
$$

a fact that follows from (III.6).

Therefore $\Phi'_{\mathscr{M}_{-n+1}}(\mathscr{F}^{\cdot})$ belongs to the triangulated category generated by E_1 and E_2, $\langle \mathrm{E}_1, \mathrm{E}_2\rangle$. By induction it follows that, for all r, $\Phi'_{\mathscr{M}_{-n+r}}(\mathscr{F}^{\cdot})$ belongs to $\langle \mathrm{E}_1, \dots, \mathrm{E}_{r+1}\rangle$. In conclusion

$$\mathscr{F}^{\cdot} \simeq \Phi'_{\mathcal{O}_{\Delta}}(\mathscr{F}^{\cdot}) \in \langle \mathrm{E}_1, \dots, \mathrm{E}_{n+1}\rangle = \langle \mathcal{O}(-n), \dots, \mathcal{O}\rangle .$$

for all complexes of sheaves $\mathscr{F}^{\cdot}$ in the bounded derived category $\mathbf{D}^{\mathrm{b}}\mathbb{P}$.

III.3.3 Corollary

For all $a \in \mathbb{Z}$, the sequence

$$\mathcal{O}(a), \dots, \mathcal{O}(a+n)$$

is strong full and exceptional.

Proof

The sequence is strongly exceptional. Fullness follows from noticing that $-\otimes^{\mathbf{L}} \mathcal{O}(a)$ is an exact auto-equivalence of $\mathbf{D}$, for any $a \in \mathbb{Z}$.

III.4 THE FIRST EQUIVALENCE

Following [Beĭ78] we find that $\mathbf{D}^{\mathrm{b}}\mathbb{P}(\mathrm{V})$ is equivalent to a simple homotopy category of modules over the symmetric algebra of $\mathrm{V}^{\vee}$.

III.4.1 Lemma

Let C and D be two triangulated categories; let $\mathrm{F} : \mathsf{C} \to \mathsf{D}$ be an exact functor. Let $\{\mathrm{X}_j\}_j$ be a family of objects generating C and assume $\{\mathrm{FX}_j\}_j$ generates D. Assume moreover that F induces isomorphisms

$$\mathrm{Hom}_{\mathsf{C}}(\mathrm{X}_i, \mathrm{X}_j[l]) \to \mathrm{Hom}_{\mathsf{D}}(\mathrm{FX}_i, \mathrm{FX}_j[l])$$

for all i, j and for all $l \in \mathbb{Z}$. Then F is an exact equivalence.

Proof
Consider C' the full subcategory of C whose objects are the Y such that

$$\mathrm{Hom}_{\mathsf{C}}(\mathrm{X}_i[l], \mathrm{Y}) \xrightarrow{\mathrm{F}} \mathrm{Hom}_{\mathsf{D}}(\mathrm{FX}_i[l], \mathrm{FY})$$

is an isomorphism. We notice that C' is additive, closed under shifts and extensions therefore it is a full triangulated subcategory of C. Since C' contains $\{\mathrm{X}_j\}$ it is equivalent via the inclusion to C. Let now D' be the image through F of C'; D' is additive, closed under shifts and extensions. Therefore D' is a full triangulated subcategory of D containing $\{\mathrm{FX}_j\}$ hence equivalent via the inclusion to D. Considering the following commutative diagram of functors

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathrm{F}"] & \mathsf{D} \\ \mathsf{C}' \arrow[r, "\mathrm{F}"] \arrow[u, hook] & \mathsf{D}' \arrow[u, hook] \end{tikzcd}$$

yields that F is an equivalence. Since it is a general fact that a quasi-inverse to a Δ-functor is also a Δ-functor we have our claim (see [Mur07, Lemma 49]).

Notation Let $\mathrm{S}_\bullet$ be a graded $\Bbbk$-algebra.

$\mathrm{S}_\bullet(-r)$ is the usual twist by $-r$ of Serre (i.e. $\mathrm{S}_\bullet(-r)$ is the free graded $\mathrm{S}_\bullet$-module with generator of degree r).

$\mathsf{GrS}_\bullet\text{-}\mathsf{Mod}$ is the category of graded $\mathrm{S}_\bullet$-modules with morphisms of degree zero.

$\mathsf{M}_{[0,n]}(\mathrm{S}_\bullet)$ is the full additive subcategory of $\mathsf{GrS}_\bullet\text{-}\mathsf{Mod}$ whose objects are modules isomorphic to finite direct sums of the $\mathrm{S}_\bullet(-r)$s, where $0 \le r \le n$.

$\mathbf{K}^{\mathrm{b}}_{[0,n]}(\mathrm{S}_\bullet)$ is the homotopic category of bounded complexes of

$$\mathsf{M}_{[0,n]}(\mathrm{S}_\bullet).$$

Finally, we fix a $\Bbbk$-vector space V of dimension $n+1$ and write $\mathsf{M}(\mathrm{Sym})$ and $\mathsf{M}(\bigwedge)$ for $\mathsf{M}_{[0,n]}(\mathrm{Sym}^\bullet \mathrm{V}^\vee)$ and $\mathsf{M}_{[0,n]}(\bigwedge^\bullet \mathrm{V})$ respectively; we write $\mathbf{K}(\mathrm{Sym})$ and $\mathbf{K}(\bigwedge)$ for $\mathbf{K}^{\mathrm{b}}_{[0,n]}(\mathrm{Sym}^\bullet \mathrm{V}^\vee)$ and $\mathbf{K}^{\mathrm{b}}_{[0,n]}(\bigwedge^\bullet \mathrm{V})$ respectively.
We now refine the first part of the proof of Theorem III.3.2.

III.4.2 Lemma
We have isomorphisms

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}(-i), \mathcal{O}(-j)) = \mathrm{Sym}^{i-j}(\mathrm{V}^\vee)$$

where composition of sheaf homomorphisms corresponds to multiplication in $\mathrm{Sym}^\bullet \mathrm{V}^\vee$.

Proof
This is a direct consequence of [Har77, Proposition II.5.13].

Using the preceding lemma we can define an additive functor

$$\widetilde{\mathrm{F}} : \mathsf{M}(\mathrm{Sym}) \to \mathsf{Coh}\mathbb{P}$$

such that $\widetilde{\mathrm{F}}\,(\mathrm{Sym}^{\bullet}\,\mathrm{V}^{\vee}(-r)) = \mathcal{O}(-r)$. Passing to the homotopic category and composing on the left with the quotient functor we obtain

$$\mathrm{F} : \mathbf{K}(\mathrm{Sym}) \to \mathbf{D}^{\mathrm{b}}\mathbb{P}.$$

III.4.3 Theorem (Beĭlinson)
The functor F *defined above is an exact equivalence.*

Proof
One only needs to check that F lies in the setting of Lemma III.4.1, which is straightforward. ❧

We also notice that there are 'dual' statements, involving the exterior algebra (as opposed to the symmetric algebra which we've just employed). We only state them.

III.4.4 Lemma
There are isomorphisms

$$\mathrm{Hom}_{\mathcal{O}}(\Omega^{i}(i), \Omega^{j}(j)) \simeq \bigwedge^{j-i} \mathrm{V}$$

where composition of sheaf homomorphisms corresponds to multiplication in $\bigwedge^{\bullet} \mathrm{V}$.

Therefore we may define the functors

$$\widetilde{\mathrm{F}'} : \mathsf{M}(\textstyle\bigwedge) \to \mathsf{Coh}\mathbb{P}$$
$$\mathrm{F}' : \mathbf{K}(\textstyle\bigwedge) \to \mathbf{D}^{\mathrm{b}}\mathbb{P}$$

such that $\widetilde{\mathrm{F}'}(\bigwedge^{\bullet} \mathrm{V}(-r)) = \Omega^{r}(r)$. One also has the following lemma.

III.4.5 Lemma
For $l > 0$ *the following holds.*

$$\mathrm{Ext}^{l}_{\mathcal{O}}(\Omega^{i}(i), \Omega^{j}(j)) = 0.$$

As a consequence have the following result, analogous to (and with analogous proof of) Theorem III.3.2.

III.4.6 Theorem
The sequence

$$\mathcal{O}, \Omega^{1}(1), \ldots, \Omega^{n}(n)$$

is strong full and exceptional.

In conclusion we obtain the following theorem.

III.4.7 Theorem (Beĭlinson)
The functor F' *constructed above is an exact equivalence.*

III.5 THE SECOND EQUIVALENCE

We now relate $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}\mathbb{P}$ to another triangulated category. We skate over some technical details and refer to [Cra08, SECTION 6] for full proofs.

Recall that a ring R has *finite global dimension* if there is an integer d such that any module admits a projective resolution of length less than d.

III.5.1 DEFINITION
Let X *be projective over* $\mathbb{k}$ *and regular. Let*

$$\mathrm{T} = \bigoplus_i \mathscr{T}_i$$

be a coherent sheaf on X. *Consider the following properties.*

T1 *The* $\mathbb{k}$*-algebra* $\mathrm{R} = \mathrm{End}_{\mathrm{X}}(\mathrm{T}, \mathrm{T})$ *has finite global dimension.*

T2 *The modules* $\mathrm{Ext}^l_{\mathrm{X}}(\mathrm{T}, \mathrm{T})$ *vanish, for* $l > 0$.

T3 *The sheaves* $\mathscr{T}_i$ *generate* $\mathbf{D}^{\mathrm{b}}\mathrm{Coh}\mathrm{X}$.

We say that T *is a* partial tilting sheaf *if it satisfies* T1 *and* T2. *The sheaf* T *is a* tilting sheaf *if it satisfies* T1 *through* T3.

The following are tilting sheaves on $\mathbb{P}$.

$$\mathrm{T} = \bigoplus_{i=0}^{n} \mathcal{O}_{\mathbb{P}}(i + a), \text{ for a fixed } a \in \mathbb{Z};$$
$$\mathrm{T}' = \bigoplus_{i=0}^{n} \Omega^i_{\mathbb{P}}(i).$$

For the theorem that follows below we need a lemma.

III.5.2 LEMMA
Let K *be a triangulated category and let* $\mathscr{A}$ *be a collection of objects which generates* K. *If* $\mathrm{E} \in \mathrm{K}$ *is an object of* K *such that*

$$\mathrm{Hom}(\mathrm{A}, \mathrm{E}[k]) = 0, \text{ for all } k \in \mathbb{Z}; \tag{III.8}$$

for all $\mathrm{A} \in \mathscr{A}$, *then* $\mathrm{E} = 0$.

Proof
Let K′ be the full subcategory consisting of all objects satisfying (III.8). Our assumptions imply that $\mathrm{K}' \supset \mathscr{A}$, and K′ is obviously closed under direct sums and translations. If $\mathrm{A} \to \mathrm{B} \to \mathrm{C} \to \mathrm{TA}$ is a d.t. with $\mathrm{A}, \mathrm{B} \in \mathrm{K}'$ then, being $\mathrm{Hom}(\text{-}, \mathrm{E}[k])$ cohomological, we obtain an exact sequence

$$0 = \mathrm{Hom}(\mathrm{TA}, \mathrm{E}[k]) \to \mathrm{Hom}(\mathrm{C}, \mathrm{E}[k]) \to \mathrm{Hom}(\mathrm{B}, \mathrm{E}[k]) = 0$$

and as a result $\mathrm{Hom}(\mathrm{C}, \mathrm{E}[k]) = 0$ for all $k \in \mathbb{Z}$. Thus K′ is a triangulated subcategory containing $\mathscr{A}$, therefore it is equivalent to the whole K. As a consequence E is isomorphic to an object of K′ and thus $\mathrm{Hom}(\mathrm{E}, \mathrm{E}) = 0$, which implies that $\mathrm{E} = 0$. ❧

The following result is due independently to Baer and Bondal ([Bae88], [Bon89]).

III.5.3 Theorem (Baer-Bondal)
Let T *be a tilting sheaf. Then the following functors are mutually quasi-inverses of each other.*

$$\mathbf{R}\mathrm{Hom}_X(T,-) : \mathbf{D}^b\mathsf{Coh}\mathbb{P} \to \mathbf{D}^b\mathsf{Mod}^f\text{-}R$$
$$-\otimes^{\mathbf{L}}_R T : \mathbf{D}^b\mathsf{Mod}^f\text{-}R \to \mathbf{D}^b\mathsf{Coh}\mathbb{P}.$$

Hence the bounded derived category of coherent sheaves on $\mathbb{P}$ *is equivalent to the bounded derived category of right finitely generated* R*-modules.*

Sketch of Proof
One starts with the functors

$$F = \mathrm{Hom}_X(T,\text{-}) : \mathsf{Qcoh}X \longrightarrow \mathsf{Mod}\text{-}R,$$
$$G = \text{-}\otimes_R T : \mathsf{Mod}\text{-}R \longrightarrow \mathsf{Qcoh}X.$$

Given a sheaf $\mathscr{F}$, $F(\mathscr{F}) = \mathrm{Hom}_X(T,\mathscr{F})$ becomes a right R-module by precomposition. For every open subset $U \subset X$ of X, $T(U)$ itself is a left R-module. Thus, given a right R-module M, one can form the tensor product $M \otimes_R T(U)$ of M and $T(U)$ over R. We then define $G(M) = M \otimes_R T$ to be the sheafification of the presheaf

$$U \longmapsto M \otimes_R T(U).$$

One sees that F and G are respectively left and right exact.

Since $\mathsf{Qcoh}X$ has enough injectives we may derive F on the right, which gives

$$\mathbf{R}F : \mathbf{D}^b\mathsf{Coh}X \longrightarrow \mathbf{D}^+\mathsf{Mod}\text{-}R$$

when restricted to bounded complexes of coherent sheaves. Given a coherent sheaf $\mathscr{F}$, one shows that the cohomology modules

$$\mathbf{R}^l F(\mathscr{F}) = \mathrm{Ext}^l_X(T,\mathscr{F})$$

vanish for big $|l|$ (and this is a consequence of the regularity of X and the local-to-global spectral sequence for Ext) and are finitely generated. As a consequence we can restrict the codomain of $\mathbf{R}F$.

$$\mathbf{R}F : \mathbf{D}^b\mathsf{Coh}X \longrightarrow \mathbf{D}^b\mathsf{Mod}^f\text{-}R.$$

On the other hand the category $\mathsf{Mod}\text{-}R$ has enough projectives, therefore G can be derived on the left. One shows that if we start with a bounded complex of finitely generated modules $M^\cdot$, the cohomology sheaves

$$\mathscr{H}^{-l}\left(M^\cdot \otimes^{\mathbf{L}}_R T\right) = \mathscr{T}\!or^R_l(M^\cdot, T)$$

vanish for big $|l|$ (as a result of R having finite global dimension) and are coherent. Hence we have a well-defined functor

$$\mathbf{L}G : \mathbf{D}^b\mathsf{Mod}^f\text{-}R \longrightarrow \mathbf{D}^b\mathsf{Coh}X.$$

Since T satisfies T2 we have

$$\mathbf{R}F \circ \mathbf{L}G(R) = \mathbf{R}F\left(R \otimes^{\mathbf{L}}_R T\right) = \mathbf{R}\mathrm{Hom}_X(T,T) = \mathrm{Hom}_X(T,T) = R.$$

This identity is the key ingredient. We want to show that $\mathbf{L}G$ fully faithful and essentilly surjective, and thus an equivalence. Let's assume for now that the above identity $\mathbf{R}F \circ \mathbf{L}G(M^{\cdot}) = M^{\cdot}$ is true for any M in $\mathbf{D}^b\mathrm{Mod}^f$-R. The functor $\mathbf{L}G$ is then fully faithful.

Let $\mathscr{E}^{\cdot}$ be an element of $\mathbf{D}^b(\mathrm{CohX})$ such that $\mathbf{R}F(\mathscr{E}^{\cdot}) = 0$. Then, if $T \cong \mathscr{S} \oplus \mathscr{T}$,

$$0 = \mathbf{R}F(\mathscr{E}^{\cdot}) = \mathbf{R}\mathrm{Hom}(T, \mathscr{E}^{\cdot}) \cong \mathbf{R}\mathrm{Hom}(\mathscr{S}, \mathscr{E}^{\cdot}) \oplus \mathbf{R}\mathrm{Hom}(\mathscr{T}, \mathscr{E}^{\cdot}).$$

In particular, for any direct summand $\mathscr{T}$ of T, we have

$$0 = \mathbf{R}^k\mathrm{Hom}(\mathscr{T}, \mathscr{E}^{\cdot}) = \mathrm{Hom}_{\mathbf{D}(X)}(\mathscr{T}, \mathscr{E}^{\cdot}[k]),$$

and, since T satisfies T3 and applying Lemma (III.5.2), we conclude that $\mathscr{E}^{\cdot} = 0$.

Now, let $\mathscr{T}^{\cdot}$ be a bounded complex of coherent sheaves and let $E^{\cdot}$ be one of its resolutions by locally frees. We have a map $\mathbf{L}G(\mathbf{R}F(E^{\cdot})) \to E^{\cdot}$ defined as follows: $\mathbf{L}G(\mathbf{R}F(E^{\cdot}))$ is the sheafification of the presheaf

$$U \longmapsto \mathrm{Hom}^{\cdot}(T, E^{\cdot}) \otimes^{\mathbf{L}}_{R} T(U)$$

and by *evaluation* we define a morphism of presheaves which is then carried over to a morphism $\xi : \mathbf{L}G(\mathbf{R}F(E^{\cdot})) \to E^{\cdot}$. Taking cones, we have a d.t.

$$\mathbf{L}G(\mathbf{R}F(E^{\cdot})) \xrightarrow{\xi} E^{\cdot} \longrightarrow \mathscr{C} \xrightarrow{[1]} .$$

Since $\mathbf{R}F$ is a Δ-functor we obtain another d.t.

$$\mathbf{R}F(\mathbf{L}G(\mathbf{R}F(E^{\cdot}))) \xrightarrow{\mathbf{R}F(\xi)} \mathbf{R}F(E^{\cdot}) \longrightarrow \mathbf{R}F(\mathscr{C}) \xrightarrow{[1]}$$

where, since $\mathbf{R}F \circ \mathbf{L}G$ is the identity, the first object is just $\mathbf{R}F(E^{\cdot})$ and one can check that $\mathbf{R}F(\xi)$ is the identity, which is an isomorphism. Therefore, by Lemma I.2.14, $\mathbf{R}F(\mathscr{C}) = 0$, and thus $\mathscr{C} = 0$. Again, by Lemma I.2.14, ξ is an isomorphism, hence $E^{\cdot}$ lies in the essential image of $\mathbf{L}G$, hence proving that $\mathbf{L}G$ is essentially surjective.

Thus it only remains to prove that the identity $\mathbf{R}F \circ \mathbf{L}G(R) = R$ can be extended to the whole of $\mathbf{D}^b\mathrm{Mod}^f$-R. It is obviously true that it holds for any free module. If we prove that it furthermore holds for any projective module of finite rank then, since R has finite global dimension, it will hold for any bounded complex of finitely generated modules. We use the fact (see [Cra08]) that the projectives of R have a very simple form: they all come from direct summands of T, under the functor F. If $T \simeq T_1 \oplus T_2$, we call $R_i = \mathrm{Hom}(T, T_i)$, which is a direct summand of R, and thus projective. We want to show that $\mathbf{L}G(R_j) = G(R_j) = T_j$. Using the *evaluation* map constructed above we get a sheaf homomorphism

$$R_j \otimes_R T \longrightarrow T_j,$$

and we want to show that it is in fact an isomorphism. Consider the endomorphisms 1_j of T, defined as being the identity on T_j and zero on the other direct summand. This endomorphism acts on $R_i \otimes_R T$. Take an element $r_i \otimes (t_i + t_j)$:

$$r_i \otimes t_j = r_i \otimes 1_j \cdot (t_i + t_j) = r_i \cdot 1_j \otimes (t_i + t_j) = \delta_{ij} r_i \otimes (t_i + t_j),$$

thus $R_i \otimes_R T = T_i$. Since any projective is of the form R_j above, we have $\mathbf{R}F \circ \mathbf{L}G(R_j) = \mathbf{R}F(T_j) = R_j$. ❧

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COLOPHON

This document has, of course, been typeset with LaTeX, using the Classic-Thesis style. The diagrams were drawn using the TikZ package.
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