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Relative Singular Twisted Bondal–Orlov

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Abstract. A theorem of Bondal and Orlov states that a smooth projective variety with ample or anti-ample canonical bundle can be reconstructed from its derived category. Here we extend the original result in three directions: we allow Gorenstein singularities, we consider derived categories of twisted sheaves and work relatively over a base stack.

1. Introduction

In a way, the starting point of the present work is a famous theorem of Gabriel [15], which says that two varieties X, Y are isomorphic if and only if their categories of coherent sheaves are equivalent, $\mathtt{Coh}(X) \simeq \mathtt{Coh}(Y)$. This theorem has seen a number of generalizations [27, 25, 1, 11, 10] (see also [3, 20, 7, 8] for what happens when you take the tensor structure into account).

When passing to derived categories, it is well known that the analogue of Gabriel's theorem is false: the most famous example probably being abelian varieties [22]. However, a foundational result of Bondal and Orlov says that at the very ends of the Kodaira dimension spectrum a derived analogue of Gabriel's theorem does indeed hold.

Theorem (Bondal–Orlov [9]) – Let X and Y be smooth projective varieties over a field k. Assume the canonical bundle ω_X is either ample or anti-ample. Then, $X \simeq Y$ (over k) if and only if $\mathtt{D}^b\mathtt{Coh}(X) \simeq \mathtt{D}^b\mathtt{Coh}(Y)$, as graded k-linear categories. ⋆

Here $\mathtt{D}^b\mathtt{Coh}(X)$ is the category of *bounded coherent* complexes. Since X is smooth, $\mathtt{D}^b\mathtt{Coh}(X)$ coincides with $\mathtt{Perf}(X)$, the category of *perfect* complexes. We write $\mathtt{D}(X)$ for the *unbounded quasi-coherent* derived category.

The theorem can be generalized to a relative setting while also incorporating some singularities.

Theorem A (6.2) – Let S be a noetherian Artin stack with affine diagonal and let $X, Y \to S$ be flat, proper and relative algebraic spaces. Assume also that for all $s \in S$, the fibres X_s, Y_s are projective, connected and Gorenstein and that X_s has either ample or anti-ample canonical bundle. Then $X \simeq Y$ as S-stacks if and only if there exists an S-linear Fourier–Mukai equivalence $\mathtt{D}(X) \simeq \mathtt{D}(Y)$. ⋆

More generally, we can work with twisted sheaves.

Theorem B (8.9) – With the same assumptions as above, let α, β be Brauer classes on X and Y. Then, there exists an isomorphism $f\colon X \to Y$ of S-stacks, such that $f^*\beta = \alpha$, if and only if $\mathtt{D}(X,\alpha) \simeq \mathtt{D}(Y,\beta)$. ⋆

In [9], the authors also showed that the group of exact auto-equivalences of $\mathtt{D}^b\mathtt{Coh}(X)$ is made up of the so-called *standard* equivalences.

Theorem – Let X be a smooth, connected, projective variety over a field k, with either ample or anti-ample canonical bundle. Then

$$\mathrm{Aut}_k(\mathtt{D}^{\mathrm{b}}\mathtt{Coh}(X)) \simeq \mathbf{Z} \times (\mathrm{Aut}_k(X) \ltimes \mathrm{Pic}(X))$$

where $\mathbf{Z}$ acts by shifts. ⁎

Here we show a parallel result.

Theorem C (7.1, 8.11) – Let $X \to S$ be as in Theorem A and assume moreover S to be connected. Then all S-linear Fourier–Mukai auto-equivalences of $\mathtt{D}(X)$ are standard:

$$\mathrm{Aut}_S(\mathtt{D}(X)) \simeq \mathbf{Z} \times (\mathrm{Aut}_S(X) \ltimes \mathrm{Pic}(X)).$$

A similar result holds for $\mathtt{D}(X,\alpha)$. We have $\mathrm{Aut}_S(\mathtt{D}(X,\alpha)) \cong \mathbf{Z} \times \mathrm{Aut}_S^{\mathbf{BG}_m}(\mathcal{X}_\alpha)$, where $\mathcal{X}_\alpha$ is the total stack of the $\mathbf{G}_m$-gerbe α, and $\mathrm{Aut}_S^{\mathbf{BG}_m}(\mathcal{X}_\alpha)$ denotes the group of $\mathbf{BG}_m$-equivariant automorphisms. ⁎

• **A few remarks**. Recall that an S-*linear integral transform* is a functor of the form $q_*(p^*(-) \otimes K)$ for an object $K \in \mathtt{D}(X \times_S Y)$ (called the *kernel*), where p and q are the projections from $X \times_S Y$ and all functors are implicitly derived. We say that an equivalence is *Fourier–Mukai* if it is (isomorphic to) an integral transform. In this article we assume all our equivalences are Fourier–Mukai transforms.

We do not feel especially guilty about this assumption. Indeed, a fundamental result of Orlov says that for smooth projective varieties *all* derived equivalences are given by integral transforms [23]. While this is not true in general (and there are many articles devoted to studying these exotic functors, see for example [12] and references therein) this becomes true if one works with *enhancements* [32, 4] (see also [13, 19]). By enhancement we mean viewing derived categories as dg or ∞-categories.

In the theorems above we use the "big" category $\mathtt{D}(X)$ rather than $\mathtt{Perf}(X)$ or the *bounded coherent* derived category $\mathtt{D}^{\mathrm{b}}\mathtt{Coh}(X)$. Since we are not imposing any condition on our base S (other than noetherianness and affineness of the diagonal) and given the possible singularities of the fibres, the former seems a natural choice.

The category $\mathtt{Perf}(X)$ can be characterized as the subcategory of *compact* objects of $\mathtt{D}(X)$. Thus, $\mathtt{D}(X) \simeq \mathtt{D}(Y)$ implies $\mathtt{Perf}(X) \simeq \mathtt{Perf}(Y)$. Conversely (working with enhancements), $\mathtt{D}(X)$ is the Ind-completion of $\mathtt{Perf}(X)$. So $\mathtt{Perf}(X) \simeq \mathtt{Perf}(Y)$ implies $\mathtt{D}(X) \simeq \mathtt{D}(Y)$.

Finally, when S is a field, one can characterize $\mathtt{D}^{\mathrm{b}}\mathtt{Coh}(X)$ as a sort of dual of $\mathtt{Perf}(X)$. In particular, if $\mathtt{Perf}(X) \simeq \mathtt{Perf}(Y)$ then $\mathtt{D}^{\mathrm{b}}\mathtt{Coh}(X) \simeq \mathtt{D}^{\mathrm{b}}\mathtt{Coh}(Y)$. This point of view is explained in [28, 2, 5]. However, when S is an arbitrary base, the situation appears to be quite subtle. See [5] for results in this direction.

To remedy this, we will assume that our equivalences are defined by kernels which lie in $\mathtt{D}^{-}\mathtt{Coh}(X \times_S Y)$, the *bounded above coherent* derived category, and that their inverses also lie in $\mathtt{D}^{-}\mathtt{Coh}(X \times_S Y)$. The assumption on the inverse is likely to be automatic. See for example [26, Corollary 1.2].

• **Relation to earlier work**. In [9], the case of smooth and projective varieties over a field is treated, hence in this context $\mathtt{Perf} = \mathtt{D}^{\mathrm{b}}\mathtt{Coh}$. The authors define the notions of *point* and *invertible* objects of $\mathtt{D}^{\mathrm{b}}\mathtt{Coh}(X)$. They then show that for X with ample or anti-ample canonical bundle these correspond precisely to (shifts of) skyscrapers and line bundles. Using the intrinsic nature of the Serre functor (which, up to a shift, is given by tensoring with the canonical) they are then able to reconstruct the canonical (or anti-canonical) graded ring of X, which is enough to conclude. Once more, we mention that $\mathtt{D}^b\mathtt{Coh}(X)$ is considered only with its graded structure and Fourier–Mukai transforms are not used. The authors then describe the group of *exact* auto-equivalences of $\mathtt{D}^b\mathtt{Coh}(X)$, viewing the latter as a *triangulated* category.

In [29, Theorem 4.19], Rouquier gave a different proof analyzing thick subcategories. This proof is similar in spirit to Gabriel's original approach to reconstructing a variety X from the category $\mathtt{Coh}(X)$.

In [24] the theorem is proved using the existence of Fourier–Mukai transforms (see also [16, Proposition 6.1, Exercise 6.2]).

In [2], the theorem is extended to Gorenstein singularities. The proof is based on a characterization of $\mathtt{D}^b\mathtt{Coh}(X)$ as a particular category of functors $\mathtt{Perf}(X) \to \mathtt{Vect}_k$ due to Rouquier [28, Proposition 6.12]. As an application, Ballard deduces the Gorenstein version of the Bondal–Orlov theorem and a description of all exact auto-equivalences.

In [21], the case of twisted sheaves on a smooth projective variety was considered. The proof parallels [16, Proposition 6.1].

In [31], the authors consider a relative setup. In order to deal with singularities, they introduce a variant of the notion of point object. They use so-called *Gorenstein cycles*, which are appropriately thickened versions of skyscraper sheaves. They assume that S is both a scheme and Cohen-Macaulay. On the other hand, they do not need their spaces to be flat over S, but only Tor-finite.

The present article provides a moduli theoretic proof of the Bondal–Orlov theorem. Given $\mathtt{D}(X)$, we construct a moduli space BO_X parameterizing what we call ***Bondal–Orlov points***, which are the natural relative version of the point objects of [9]. We mention in passing that we never use the notion of invertible object introduced in [9]. Using the machinery developed in [11], we are then able to compare BO_X with the functor of points of X. With additional work, the theorems follow.

Contents

2. Conventions

A *point* z of a stack Z will mean any morphism $z\colon \operatorname{Spec} k \to Z$, where k is a field. We write $z \in Z$. This is slightly non-standard, for example in [17] a point means an element of the underlying topological space $|Z|$.

In the sequel we will fix a base Artin stack S, which will be assumed to be noetherian and with affine diagonal. We will want to define prestacks over S, which for us means considering (weak) functors

$$F\colon \mathtt{Aff}_{\mathrm{ft}}^{\mathrm{op}}/S \to \mathtt{Grpd}.$$

The subscript $_{\mathrm{ft}}$ stands for "finite type". Concretely, F assigns a groupoid to any finite type morphism $\operatorname{Spec} R \to S$. This is justified by the fact that we are only interested in stacks locally of finite type over S, whose functor of points on the whole $\mathtt{Aff}^{\mathrm{op}}/S$ are determined by restricting to $\mathtt{Aff}_{\mathrm{ft}}^{\mathrm{op}}/S$. In particular, since S is noetherian, all the rings we consider will be noetherian.

We say that a morphism $F \to G$ is *fully faithful on objects* if, for all $\operatorname{Spec} R \to S$, the functor $F(R) \to F(R)$ is fully faithful.

We say that $X \to S$ is a *relative (insert whatever)* if for any $\operatorname{Spec} R \to S$ the fibre product $X_R = X \times_S \operatorname{Spec} R$ is an (insert whatever).

We will use the following notation. We write $\mathtt{Coh}(X)$, $\mathtt{QCoh}(X)$ for the abelian categories of coherent and quasi-coherent sheaves on X. By $\mathtt{D}(X)$ we mean the *unbounded* derived category of *quasi*-coherent sheaves. We have inclusions

$$\mathtt{Perf}(X) \subset \mathtt{D}^b\mathtt{Coh}(X) \subset \mathtt{D}^-\mathtt{Coh}(X) \subset \mathtt{D}(X)$$

where the first is the category of perfect complexes, the second is the *bounded coherent* derived category and the third is the category of *bounded above and coherent* complexes. As all our X are noetherian, a bounded (resp. bounded above) complex $E \in \mathtt{D}(X)$ lies in $\mathtt{D}^b\mathtt{Coh}$ (resp. $\mathtt{D}^-\mathtt{Coh}$) if and only if $H^i(E) \in \mathtt{Coh}(X)$ for all $i \in \mathbf{Z}$.

Let X, Y be stacks over a base S. By S*-linear Fourier–Mukai equivalence* we mean a functor $\mathtt{D}(X) \to \mathtt{D}(Y)$ represented by a kernel $K \in \mathtt{D}^-\mathtt{Coh}(X \times_S Y)$ whose inverse is also represented by a kernel in $\mathtt{D}^-\mathtt{Coh}(X \times_S Y)$. By $\operatorname{Aut}_S(\mathtt{D}(X))$ we mean the group of S-linear Fourier–Mukai equivalences.

Following Section 4, all functors are implicitly derived.

3. Points in abelian categories

Here we mimic the approach taken in [11]. The idea being that a variety X can be realized as a moduli of point-like objects of $\mathtt{QCoh}(X)$.

Situation 3.1

Let S *be a noetherian Artin stack with affine diagonal. Let* $\pi\colon X \to S$ *be a relative algebraic space. Assume* π *is proper.* $\ast$

For any affine $\operatorname{Spec} R \to S$ we write X_R for the base change $X \times_S \operatorname{Spec} R$. For a quasi-coherent sheaf $M \in \mathtt{QCoh}(X)$ we write $M \otimes_S R$ (or $M \otimes R$ if no confusion is likely to arise) for the pullback along the projection $X_R \to X$. We draw the following diagram for later reference.

$$\begin{tikzcd} X_R = X \times_S \operatorname{Spec} R \arrow[r] \arrow[d, "\pi_R"] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] & S \end{tikzcd}$$

In [11] there is a notion of point-like objects of the category $\mathtt{QCoh}(X)$. Since here our base S is noetherian and X is proper over it, we may give a simplified version of [11, Definition 2.1] (in loc. cit., see also Remark 2.3 and the comment just before Section 2.2). We will show that it leads to the same moduli problem below.

Definition 3.2 – Assume to be working in Situation 3.1. For any $\operatorname{Spec} R \to S$ of finite type, let X_R be the base change. A $P \in \mathtt{QCoh}(X_R)$ is *lazily point-like* if the following hold.

(1) P is finitely generated.
(2) P is R-flat.
(3) The natural morphism $R \to \operatorname{Hom}(P,P)$ is an isomorphism.
(4) If R is a field then P has no non-trivial quotients.

Such P is moreover *universally* lazily point-like if for any $R \to R'$ the base change $P \otimes_R R' \in \mathtt{QCoh}(X_{R'})$ is lazily point-like. *

One then defines Pt_X as the moduli functor parameterizing universally lazily point-like sheaves.

Remark 3.3. Notice that, if $P \in \mathrm{Pt}_X(R)$ and L is a line bundle on Spec R, then $P \otimes \pi_R^* L$ is still an object of $\mathrm{Pt}_X(R)$. Said in a more compact way, $\mathbf{BG}_m$ acts on Pt_X. *

In parallel with [11, Theorem 3.2], let us show that Pt_X is isomorphic to $X \times \mathbf{BG}_m$, where the isomorphism is compatible with the action of $\mathbf{BG}_m$ on both sides. Notice that, for any $\operatorname{Spec} R \to S$, the groupoid $X \times \mathbf{BG}_m(R)$ consists of pairs (f, L) where $f\colon \operatorname{Spec} R \to X$ is a morphism (over S) and L is a line bundle on R. There is an obvious morphism $X \times \mathbf{BG}_m \to \mathrm{Pt}_X$ which on R-points sends a pair (f, L) to $\mathcal{O}_{\Gamma_f} \otimes \pi_R^* L$, where $\Gamma_f \subset X \times \operatorname{Spec} R$ is the graph of f.

Proposition 3.4 – Assume to be working in Situation 3.1. The map $X \times \mathbf{BG}_m \to \mathrm{Pt}_X$ is an isomorphism of S-stacks. *

Proof. The proof boils down to the following claim: given $\operatorname{Spec} R \to S$ and a lazily point-like sheaf $P \in \mathtt{QCoh}(X \times_S \operatorname{Spec} R)$ there exists a unique morphism $f\colon \operatorname{Spec} R \to X$ (over S) and a unique line bundle L on Spec R such that $P \otimes L^{-1}$ is isomorphic to $\mathcal{O}_{\Gamma_f}$.

In what follows we go through the proof in [11] for point-like sheaves and explain where the extra axioms are not needed.

Consider the scheme-theoretic support Z of P.

$$Z = \operatorname{supp} P \subset X \times \operatorname{Spec} R.$$

Write $\rho\colon Z \to \operatorname{Spec} R$ for the projection. We wish to show two things: $\pi_{R*}P$ is a line bundle; ρ is an isomorphism. The morphism f will then be given by the composition $\operatorname{Spec} R \to Z \to X \times_S \operatorname{Spec} R \to X$; the line bundle L will be given by $(\pi_R^* \pi_{R*} P)^\vee$.

If R is a field, we may apply axiom (4), so that [11, Lemma 2.7] tells us ρ is an isomorphism. For general R, the fact that P is *universally* lazily point-like implies ρ is a universal bijection with affine fibres.

Next, we want to show ρ is affine. By [30, Theorem 8.5] it suffices to show ρ is universally closed. In [11] we did not have any notherian assumption and needed [11, Definition 2.1 axiom (5)]. Here, however, this is automatic thanks to properness of π. Hence, ρ is affine.

Let us now show $\mathrm{L} := \pi_{\mathrm{R},*}\mathrm{P}$ is a line bundle. In [11, p. 10], this required [11, Definition 2.1 axiom (3)] as we needed to know L was finitely presented. But this is a consequence of properness of π, and we may proceed as in [11, p. 9] to show that L is a line bundle.

To conclude, we only need axiom (3) of being lazily point-like (which is [11, Definition 2.1 axiom (4)] in the case M = R). We proceed as in [11, p. 10] and conclude. ∎

Remark 3.5. As is transparent from the proof, Pt_X is isomorphic to $\mathrm{X} \times \mathbf{BG}_\mathrm{m}$ compatibly with action of tensoring with line bundles coming from the base. In other words, this is an isomorphism of $\mathbf{G}_\mathrm{m}$-gerbes. Informally, we say it is $\mathbf{BG}_\mathrm{m}$*-equivariant*. *

Remark 3.6. In place of Pt_X, we could consider the `Set`-valued functor, where we identify objects $\mathrm{E}_1, \mathrm{E}_2 \in \mathrm{Pt}_\mathrm{X}(\mathrm{R})$ if there exists a line bundle L on R and an isomorphism $\mathrm{E}_1 \simeq \mathrm{E}_2 \otimes \mathrm{L}$. This is the $\mathbf{G}_\mathrm{m}$-rigidification of Pt_X and we denote it by $\mathrm{Pt}_\mathrm{X}/\mathbf{BG}_\mathrm{m}$. The latter is isomorphic to X itself. Nevertheless, keeping track of automorphisms is crucial later when dealing with twisted sheaves. *

4. Base change

Before moving on, we must deal with a few technical facts. What follows is standard.

Notation. Henceforth, all functors will be implicitly derived.

The first lemma we need is a generalization of the well-known [16, Lemma 3.31].

Lemma 4.1 – Let S be a noetherian and connected scheme. Let X be a noetherian Artin stack and $f\colon \mathrm{X} \to \mathrm{S}$ be a flat map.

Let E be a bounded above complex with coherent cohomology. For any point $s \in \mathrm{S}$, denote by $i_s\colon \mathrm{X}_s \to \mathrm{X}$ the inclusion of the fibre. Assume that for all $s \in \mathrm{S}$ the derived restriction $i_s^*\mathrm{E}$ is concentrated in a single degree. Then E is a (possibly shifted) sheaf flat over S. *

Proof. Assume first S is the spectrum of a local ring with closed point $s \in \mathrm{S}$. Let $\mathrm{U} \to \mathrm{X}$ be a faithfully flat map with U affine. Notice that the claim holds (namely that E is concentrated in a single degree and flat) if and only if it holds for the pullback of E to U. Hence we may assume directly that X itself is affine. Moreover, the claim holds if it holds for the restriction of E to each of the local rings of X. Hence we may assume X to be the spectrum of a local noetherian ring and the morphism $\mathrm{X} \to \mathrm{S}$ to be local. Let us momentarily write i for the inclusion i_s of the fibre over s.

If $\mathrm{E} = 0$ there is nothing to prove. If $\mathrm{E} \neq 0$ then by assumption there exists m such that $\mathrm{H}^m(\mathrm{E}) \neq 0$ and $\mathrm{H}^{m+j}(\mathrm{E}) = 0$ for all $j > 0$. Up to shifts, we may assume that $i^*\mathrm{E}$ is concentrated in degree zero. Consider the spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(i^*\mathrm{H}^q(\mathrm{E})) \Rightarrow \mathrm{H}^{p+q}i^*(\mathrm{E}).$$

Notice that the differentials involving the $\mathrm{H}^0(i^*\mathrm{H}^m(\mathrm{E}))$ term are both zero. If $m > 0$ or $m < 0$ then $\mathrm{H}^0(i^*\mathrm{H}^m(\mathrm{E})) = \mathrm{H}^m(i^*\mathrm{E}) = 0$ hence $\mathrm{H}^m(\mathrm{E}) = 0$, which we assumed wasn't true. Thus $m = 0$.

Now, the differentials involving the $H^{-1}(i^*H^0(E))$ term are both zero. Thus, $H^{-1}(i^*H^0(E))$ is an associated graded piece of the filtration on $H^{-1}(i^*E) = 0$. Since (by assumption) the latter is zero, the former must also vanish. By the local criterion for flatness [14, TAG 00MK], we have that $H^0(E)$ is flat over S (here we use that E has coherent cohomology).

Let's move up to the case where S is an irreducible scheme. By assumption, for each $s \in S$, there exists an integer m_s such that i_s^*E is concentrated in the single degree m_s.

If s is any point of S, write $X'_s = \operatorname{Spec} \mathcal{O}_{S,s} \times_S X$. If η is the generic point of S, then we have a morphism of local rings $\mathcal{O}_{S,s} \to \mathcal{O}_{S,\eta}$ inducing a morphism $X'_\eta \to X'_s$. Since i_η^*E factors through restriction to X'_s, we have $m_s = m_\eta$. Hence E is globally concentrated in degree m_η and flat over S (as per the local discussion above).

Assume now S is a connected scheme. By the noetherian assumption, it has finitely many irreducible components. By the argument above, m_s is constant on irreducible components. By connectedness, we have that m_s is constant. ∎

For our second lemma, we change context. Let now X be a projective, connected, Gorenstein scheme. The category $\mathtt{Perf}(X)$ has a Serre functor $\Sigma = \Sigma_X$, given by $- \otimes \omega_X[\dim X]$. Moreover, Σ extends to an auto-equivalence of the whole category $\mathtt{D}(X)$.

The converse is also true: if $\mathtt{Perf}(X)$ has a Serre functor then X is Gorenstein [2, Lemma 6.6]. By a standard argument, we have the following (cfr. [16, Lemma 1.30]).

Lemma 4.2 – Let X and Y be projective, connected Gorenstein schemes over k and let Φ be an equivalence between $\mathtt{D}(X)$ and $\mathtt{D}(Y)$. Assume X is Gorenstein. Then, Y is Gorenstein and there is an isomorphism $\Sigma_Y\Phi \simeq \Phi\Sigma_X$. $*$

For the rest of this section we will work in the following setup.

Situation 4.3

> *Let* S *be a noetherian base Artin stack with affine diagonal. Let* $X, Y \to S$ *be flat and proper relative algebraic spaces with affine diagonal. Let* $K \in D^-Coh(X \times_S Y)$ *and let* $\Phi_K \colon D(X) \to D(Y)$ *be the associated integral transform. Assume* Φ_K *is an equivalence. Assume moreover that there exist a kernel* $H \in D^-Coh(X \times_S Y)$*, defining the integral transform* $\Psi_H \colon D(Y) \to D(X)$*, and an isomorphism* $\Psi_H \simeq \Phi_K^{-1}$. $*$

The following is another standard fact.

Lemma 4.4 – Assume to be working in Situation 4.3. Let $T \to S$ with T a scheme. Let $K_T, H_T \in \mathtt{D}^-\mathtt{Coh}(X_T \times_T Y_T)$ be the base changes of the respective kernels. Then the corresponding integral transforms Φ_{K_T}, Ψ_{H_T} are equivalences and $\Phi_{K_T}^{-1} = \Psi_{K_T}$. $*$

Lemma 4.5 – Assume to be working in Situation 4.3. Moreover, assume $S = \operatorname{Spec} R$ to be affine. Then, $E \in \mathtt{D}(X)$ is bounded above (below) if and only if $\Phi_K(E)$ is bounded above (below). $*$

Proof. This is [5, Lemma 3.0.14] (notice they use *connective* to mean concentrated in cohomological degrees ≤ 0). ∎

Lemma 4.6 – Assume to be working in Situation 4.3. Moreover, assume S to be affine. Let $E \in \mathtt{D}(X)$ be bounded above. Then, $E \in \mathtt{D}(X)$ has coherent cohomology if and only if $\Phi_K(E)$ has coherent cohomology. $*$

Proof. This boils down to the fact that for proper f, f_* preserves $\mathtt{D}^{-}\mathtt{Coh}$. ∎

Combining the previous lemmas we obtain the final result of this section.

Proposition 4.7 – Assume to be working in Situation 4.3. Let $T = \operatorname{Spec} R \to S$. Then Φ_{K_T} restricts to an equivalence $\mathtt{D}^b\mathtt{Coh}(X_T) \simeq \mathtt{D}^b\mathtt{Coh}(Y_T)$. ⁎

5. Points in derived categories

Now we study our moduli of Bondal–Orlov points.

Situation 5.1

Assume Situation 4.3. Moreover, assume that for all $s \in S$ the fibres X_s, Y_s are connected and Gorenstein. ⁎

By the discussion from the previous section, if X_s is Gorenstein so is Y_s.

Definition 5.2 – Assume to be working in Situation 5.1. Let $\operatorname{Spec} R \to S$ be of finite type. We say that $P \in \mathtt{D}(X_R)$ is a *Bondal–Orlov point* if the following hold.

(1) $P \in \mathtt{D}^b\mathtt{Coh}(X_R)$.
(2) The natural map $R \to \operatorname{Hom}(P, P)$ is an isomorphism.
(3) For all $i < 0$, $\operatorname{Hom}(P, P[i]) = 0$.
(4) If R is a field, there exists $m \in \mathbf{Z}$ and an isomorphism $\Sigma P \simeq P[m]$.

Finally, we say that such a P is moreover a *universal* Bondal–Orlov point if for any $R \to R'$, $P \otimes_R R'$ is a Bondal–Orlov point. ⁎

We can define a prestack

$$\mathrm{BO}_X \colon \mathtt{Aff}_{ft}^{op}/S \to \mathtt{Grpd}$$

sending a map $\operatorname{Spec} R \to S$ to the groupoid of universal Bondal–Orlov points over R.

Remark 5.3. We can parallel Remark 3.5 for BO_X. Indeed, BO_X comes with an action given by tensoring with line bundles on the base. Moreover, the constant sheaf (over S) $\mathbf{Z}$ also acts on BO_X by shifts. We will be interested in the quotient $\mathrm{BO}_X/\mathbf{Z}$. Assume $\operatorname{Spec} R \to S$ with $\operatorname{Spec} R$ connected. Concretely, objects of $\mathrm{BO}_X/\mathbf{Z}(R)$ are given by identifying $E_1, E_2 \in \mathrm{BO}_X(R)$ if $E_1 \simeq E_2[r]$ for some r. Morphisms are given by isomorphisms between E_1 and $E_2[r]$. Notice that, if $E_1 \simeq E_2[r]$, there are no morphisms in $\mathrm{BO}_X(R)$ between E_1 and $E_2[i]$, for $i \neq r$. This is because a bounded complex cannot be isomorphic to a shift of itself.

We can take this a step further and rigidify $\mathrm{BO}_X/\mathbf{Z}$ to obtain $\mathrm{BO}_X/\mathbf{Z}/\mathbf{BG}_m$. Concretely, we identify complexes $E_1, E_2 \in \mathrm{BO}_X(R)$, if there exists an integer r and a line bundle L on $\operatorname{Spec} R$, such that $E_1 \simeq E_2 \otimes L[r]$.

To deal with non-connected $\operatorname{Spec} R$, we must allow $\mathbf{Z}$ to act by separate shifts on each connected component of $\operatorname{Spec} R$. *

Proposition 5.4 – Assume to be working in Situation 5.1. Then Φ_K induces an isomorphism $\mathrm{BO}_X \simeq \mathrm{BO}_Y$ of S-stacks. This isomorphism is equivariant with respect to $(\mathbf{Z} \times \mathbf{BG}_m)$, hence induces an isomorphism $\mathrm{BO}_X/\mathbf{Z} \simeq \mathrm{BO}_Y/\mathbf{Z}$ and $\mathrm{BO}_X/\mathbf{Z}/\mathbf{BG}_m \simeq \mathrm{BO}_Y/\mathbf{Z}/\mathbf{BG}_m$. ⁎

Proof. It suffices to show that, for any $\operatorname{Spec} R \to S$ of finite type and any $E \in \mathrm{BO}_X(R)$, $\Phi_{K_R}(E) \in \mathrm{BO}_Y(R)$.

Let us check that $\Phi_{K_R}(E)$ satisfies all axioms of being a Bondal–Orlov point. (1) is true by Lemma 4.7. (2) is true as Φ_{K_R} preserves the R-module structure on Hom spaces. (3) is true as Φ_{K_R} is an equivalence. (4) is true as Φ_{K_R} commutes with Serre functors, by Lemma 4.2.

The compatibility with $\mathbf{BG}_m \times \mathbf{Z}$ follows as Φ is exact and linear over S. ■

Finally, we begin to compare Pt_X with BO_X. The following lemma will be crucial below.

Lemma 5.5 – Assume to be working in Situation 5.1. There is an obvious map $\mathrm{Pt}_X \to \mathrm{BO}_X$, which is: fully faithful on objects, formally smooth and equivariant with respect to the $\mathbf{BG}_m$-action on either side. The same holds for the composition $\mathrm{Pt}_X \to \mathrm{BO}_X/\mathbf{Z}$. Finally, the induced map $X \to \mathrm{BO}_X/\mathbf{Z}/\mathbf{BG}_m$ is a formally smooth monomorphism. ⋆

Proof. Let $\operatorname{Spec} R \to S$ be of finite type. We can check directly that $P \in \mathrm{Pt}_X(R)$ satisfies the axioms of being a Bondal–Orlov point.

- (1) follows from (1) of being lazily point-like.
- (2) follows from (3) of being lazily point-like.
- (3) follows as P is a sheaf.
- (4) follows from the fact that, over a field, the Serre functor is given by tensoring with a line bundle and point-like sheaves are skyscrapers.

Clearly everything is compatible with base change, hence we have a well defined morphism $\mathrm{Pt}_X \to \mathrm{BO}_X$. Since two sheaves are isomorphic if and only if they are quasi-isomorphic as complexes, $\mathrm{Pt}_X \to \mathrm{BO}_X$ is fully faithful on objects. Equivariance with respect to $\mathbf{BG}_m$ is also automatic: for $E \in \mathrm{Pt}_X(R)$, the derived tensor product $E \otimes L$ coincides with the underived one (since we are assuming L locally free). Let us show $\mathrm{Pt}_X \to \mathrm{BO}_X$ is formally smooth.

We need to show that any lifting problem

$$\begin{array}{ccc} \operatorname{Spec} R_0 & \xrightarrow{E} & \mathrm{Pt}_X \\ \downarrow & & \downarrow \\ \operatorname{Spec} R & \xrightarrow{G} & \mathrm{BO}_X \end{array}$$

with $R \to R_0$ a nil-thickening of Artin rings, has a solution $\operatorname{Spec} R \to \mathrm{Pt}_X$. Concretely, $G \otimes R_0 \simeq E$. In particular, the fibre of G is a sheaf and hence by Lemma 4.1, G itself is a sheaf flat over R. By inspection, the axioms of being lazily point-like are satisfied. Hence $G \in \mathrm{Pt}_X(R)$ is itself the solution to the lifting problem.

The rest of the theorem follows. ■

6. Bondal–Orlov

We now combine the previous sections to prove our main result.

Situation 6.1

Assume to be working in Situation 5.1. Moreover, for all $s \in S$, assume the canonical bundle of the fibre ω_{X_s} to be either ample or anti-ample. ⋆

Here is our main result.

Theorem 6.2 – Assume to be working in Situation 6.1. Then $X \simeq Y$ over S. ⋆

We argue as follows. We will show that (given the assumptions on the canonical) the morphism $\mathrm{Pt}_X \to \mathrm{BO}_X \to \mathrm{BO}_X/\mathbf{Z}$ is an isomorphism. Then we inspect the map $\mathrm{Pt}_Y \to \mathrm{BO}_Y/\mathbf{Z} \simeq \mathrm{BO}_X/\mathbf{Z} \simeq \mathrm{Pt}_X$ and prove the induced map $Y \to X$ is in fact an isomorphism.

Proposition 6.3 – Assume to be working in Situation 6.1. Let $\operatorname{Spec} R \to S$ be of finite type, with $\operatorname{Spec} R$ connected. If $E \in \mathrm{BO}_X(R)$ is a universal Bondal–Orlov point, there exists $r \in \mathbf{Z}$, such that $H^i(E) = 0$ for all $i \neq r$ and $H^r(E) \in \mathrm{Pt}_X(R)$ is universally lazily point-like. $*$

Proof. What follows is essentially half of the original argument of Bondal–Orlov. Let $\operatorname{Spec} R \to S$ and let P be a (universal) Bondal–Orlov point over R. Let now $s\colon \operatorname{Spec} k \to \operatorname{Spec} R$ be any point. Write ω for the canonical bundle of X_s. Since the fibre s^*P is also a Bondal–Orlov point, axiom (4) implies $s^*P \otimes \omega[d] = s^*P[n]$ for some integer n (here d is the dimension of X_s). By exactness, $H^i(s^*P) \otimes \omega \simeq H^i(s^*P)[n-d]$ which means $n = d$.

Suppose we are in the case where ω is ample (the anti-ample case is treated by considering the inverse of the Serre functor). Since $H^i(s^*P) \otimes \omega^j \simeq H^i(s^*P)$ (and using axioms (2) and (3)), we can appeal to [16, Lemma 4.5] to conclude s^*P is concentrated in a single degree.

By the Lemma 4.1, $P = Q[r]$ for some r, where Q is a sheaf flat over R. By inspection, Q satisfies the axioms of being a lazily point-like object. ∎

Proposition 6.4 – Assume to be working in Situation 6.1. Then the composition $\mathrm{Pt}_X \to \mathrm{BO}_X \to \mathrm{BO}_X/\mathbf{Z}$ is a $\mathbf{BG}_m$-equivariant isomorphism. It induces an isomorphism $X \to \mathrm{BO}_X/\mathbf{Z}/\mathbf{BG}_m$. $*$

Proof. It suffices to prove that for any $\operatorname{Spec} R \to S$ we have an equivalence of groupoids $\mathrm{Pt}_X(R) \to \mathrm{BO}_X/\mathbf{Z}(R)$, for all $\operatorname{Spec} R \to S$. Lemma 5.5 shows fully faithfulness. Essential surjectivity follows from Proposition 6.3, once we observe that the *sheaf* $\mathbf{Z}$ acts by different shifts on connected components of $\operatorname{Spec} R$ (see Remark 5.3). Once again, since for a sheaf E and a locally free sheaf L the derived tensor product $E \otimes L$ coincides with the underived one, the rest follows. ∎

We are ready to prove our main result.

Proof (of Theorem 6.2). Consider now the map

$$\mathrm{Pt}_Y \to \mathrm{BO}_Y \to \mathrm{BO}_Y/\mathbf{Z} \simeq \mathrm{BO}_X/\mathbf{Z} \simeq \mathrm{Pt}_X\,.$$

Since it is equivariant, it descends to a map $f\colon Y = \mathrm{Pt}_Y/\mathbf{BG}_m \to \mathrm{Pt}_X/\mathbf{BG}_m = X$ over S. Let $T \to S$ be faithfully flat, with T affine. It suffices to show that the base change $f_T\colon Y_T \to X_T$ is an isomorphism of T-schemes.

By properness of X and Y, we have that f_T is proper. Moreover, by Lemma 5.5, f_T is a monomorphism hence it is a closed immersion. Also using Lemma 5.5, we have that f_T is formally smooth. Hence, f_T is both a closed and an open immersion. To conclude, it suffices to show that it is surjective.

Let $x \in X_T$ be any point. Let $t \in T$ be its image in T. Consider the map on fibres $Y_t \to X_t$. The exact same argument as above shows it is an open and closed immersion. Since (by assumption) Y_t and X_t are connected, it is an isomorphism.

Hence, there exists $y \in Y_t$ mapping to x. In other words, $Y_T \to X_T$ it surjective and thus an isomorphism. ∎

7. Auto-equivalences

The goal of this section is to explain and prove the following.

Theorem 7.1 – Let S be a connected, noetherian Artin stack with affine diagonal. Let $X \to S$ be a flat and proper relative algebraic space. Assume that for all $s \in S$, the fibre X_s is Gorenstein, connected and with either ample or anti-ample canonical bundle.

Then the natural map $\mathbf{Z} \times (\mathrm{Aut}_S(X) \ltimes \mathrm{Pic}(X)) \to \mathrm{Aut}_S(\mathtt{D}(X))$ is an isomorphism. ⁎

The group of auto-equivalences of the derived category always contains the standard equivalences: shifts, tensoring with line bundles, automorphisms. More precisely, there is a group homomorphism

$$\mathbf{Z} \times (\mathrm{Aut}_S(X) \ltimes \mathrm{Pic}(X)) \to \mathrm{Aut}_S(\mathtt{D}(X)).$$

An S-linear auto-equivalence of $\mathtt{D}(X)$ induces, essentially by definition, an automorphism of BO_X over S. Moreover, this automorphism is of a special kind, as it commutes with the action of $\mathbf{Z}$ (shifts) and the action of $\mathbf{BG}_m$ (tensoring with pullbacks of line bundles on the base). We write $\mathrm{Aut}_S^{\mathbf{Z}\times\mathbf{BG}_m}(BO_X)$ for the subgroup of automorphisms commuting with shifts and tensoring with line bundles coming from the base. Given the assumptions on the fibres of X, Proposition 6.4 applies. It is then straightforward to see that $\mathrm{Aut}_S^{\mathbf{Z}\times\mathbf{BG}_m}(BO_X) \simeq \mathbf{Z} \times \mathrm{Aut}_S^{\mathbf{BG}_m}(\mathrm{Pt}_X)$. Since $\mathrm{Pt}_X \simeq X \times \mathbf{BG}_m$, this automorphism group is isomorphic to $\mathbf{Z} \times (\mathrm{Aut}_S(X) \ltimes \mathrm{Pic}(X))$. Thus, we have a right split short exact sequence

$$1 \to Q \to \mathrm{Aut}_S(\mathtt{D}(X)) \to \mathbf{Z} \times (\mathrm{Aut}_S(X) \ltimes \mathrm{Pic}(X)) \to 1$$

and the proof of Theorem 7.1 reduces to showing that Q is trivial.

Proof (of Theorem 7.1). Let K be a kernel representing an auto-equivalence in Q. Let us call this equivalence Φ. We will show that $K \simeq \mathcal{O}_\Delta$. Indeed, $\mathcal{O}_X \boxtimes K$ defines an auto-equivalence of $\mathtt{D}(X \times X)$. By inspection, the image $\mathcal{O}_\Delta$ under this equivalence is precisely K. However, $\mathcal{O}_\Delta$ is a family of Bondal–Orlov points over X. Since Φ acts as the identity on BO_X, it must send $\mathcal{O}_\Delta$ to itself. Hence, $K \simeq \mathcal{O}_\Delta$. ∎

8. Twisted sheaves

We now generalize all the above to the case of twisted sheaves.

8.1. Abelian points. Let us start with Section 3. Let S be a noetherian Artin stack with affine diagonal and let $X \to S$ be a proper relative algebraic space. Moreover, fix a $\mathbf{G}_m$-gerbe α on X. We have the category $\mathtt{QCoh}(X, \alpha)$ of α-twisted sheaves. As in Section 3, we have two moduli spaces of (universally) point-like and lazily point-like α-twisted sheaves. Just as in Section 3, both have an action of $\mathbf{BG}_m$ and are isomorphic S-stacks (with isomorphism compatible with the $\mathbf{BG}_m$-action). Henceforth, we will write Pt_α for the moduli space of lazily point-like α-twisted sheaves. By [11], we know Pt_α is isomorphic (equivariantly, i.e. as $\mathbf{G}_m$-gerbes) to α.

8.2. Base change. Assume α is a $\mathbf{G}_m$-gerbe on a space X. If there exists a finitely generated locally free α^{-1}-twisted sheaf F then the functor $(-) \otimes F$ sends α-twisted sheaves to ordinary sheaves. This standard trick (which we learned from [21]) is often useful for bootstrapping theorems.

Note that a $\mathbf{G}_m$-gerbe admits a locally free twisted sheaf if and only if the gerbe is representable by an Azumaya algebra [18, Proposition 3.1.2.1].

In general it is not possible to represent every $\mathbf{G}_m$-gerbe by an Azumaya algebra. However, as shown by Toën in [33], as long as X is assumed to be quasi-compact and quasi-separated, every $\mathbf{G}_m$-gerbe can be represented by a derived Azumaya algebra. That is, we know that there exists an α^{-1}-twisted perfect complex F. And we may consider $M\otimes F$ to obtain an ordinary perfect complex.

Lemma 8.1 – Let X be a quasi-projective scheme over a field and let α be a $\mathbf{G}_m$-gerbe over X. Let L be an ample (or anti-ample) line bundle. If $M \in \mathtt{D}^b\mathtt{Coh}(X,\alpha)$ satisfies $M \otimes L \simeq M$, then M has zero-dimensional support. *

Proof. Let $F \in \mathtt{Perf}(X,\alpha^{-1})$ be a compact generator of the category $\mathtt{Perf}(X,\alpha^{-1})$ (as guaranteed to exist by [33, Corollary 3.8]). We denote by $M' = M\otimes F \in \mathtt{D}^b\mathtt{Coh}(X)$. By assumption we have $M' \otimes L \simeq M'$. Since L is ample or anti-ample, we see that the Hilbert series of M' is constant, therefore M' has zero-dimensional support. Since F is a generator, we must have $\operatorname{supp} M' = \operatorname{supp} M$. ∎

Lemma 8.2 – Let X be a connected projective scheme over a field and α a $\mathbf{G}_m$-gerbe. The category $\mathtt{Perf}(X,\alpha)$ has a Serre functor if and only if X is Gorenstein. *

Proof. If X is Gorenstein, then $(-)\otimes\omega_X[\dim X]$ is a Serre functor; a proof can be found for example in [6, Lemma 1.8.1 (b,c)], where $\mathtt{D}^b\mathtt{Coh}$ must be replaced with $\mathtt{Perf}$ for the argument to apply to the Gorenstein case. Conversely, we can proceed as in [2]. Given a compactly generated k-linear triangulated category $\mathtt{T}$, Ballard introduces the notion of Rouquier functor R. He then proves [2, Lemma 5.13] that if $\mathtt{T}^c$ (the subcategory of compact objects) admits a weak Serre functor S then S must be isomorphic to R. On the other hand, [2, Example 5.12] shows that the Rouquier functor of $\mathtt{Perf}(X)$ must be given by $(-)\otimes f^!\mathcal{O}_k$. The proof found there applies verbatim to the case of twisted sheaves. In particular, we see that $f^!\mathcal{O}_k$ must be perfect and [2, Lemma 6.6] tells us X is in fact Gorenstein. ∎

We also have an analogue of Lemma 4.1.

Lemma 8.3 – Let S be an affine, noetherian and connected scheme and let $X \to S$ be flat and projective. Let α be a $\mathbf{G}_m$-gerbe on X. Let $E \in \mathtt{D}^-\mathtt{Coh}(X,\alpha)$. Assume that, for all $s \in S$, the (derived) restriction $E_s \in \mathtt{D}^-\mathtt{Coh}(X_s,\alpha_s)$ is concentrated in a single degree. Then, there exists $r \in \mathbf{Z}$, such that $E = E'[r]$ for $E' \in \mathtt{Coh}(X,\alpha)$ and E' flat over S. *

Proof. Let $g\colon X' \to X$ be an étale surjective morphism such that $g^*\alpha = 0$. It follows that E satisfies the claim if and only if g^*E is the shift of a coherent sheaf flat over S. But the latter is true by Lemma 4.1. ∎

The rest of Section 4 also goes through for twisted sheaves.

8.3. Derived points. Let us fix the setup for the rest of this paper.

Situation 8.4

Let S *be a noetherian Artin stack with affine diagonal. Let* $X, Y \to S$ *be flat and proper relative algebraic spaces. Let* α *be a* $\mathbf{G}_m$*-gerbe on* X*, and* β *a* $\mathbf{G}_m$*-gerbe on* Y*. Let* $K \in \mathtt{D}^-\mathtt{Coh}(X\times_S Y, \alpha^{-1}\boxtimes\beta)$ *and let* $\Phi_K\colon \mathtt{D}(X) \to \mathtt{D}(Y)$ *be the associated integral transform. Assume* Φ_K *is an equivalence. Assume moreover that there exist a kernel* $H \in \mathtt{D}^-\mathtt{Coh}(Y\times_S X, \beta^{-1}\boxtimes\alpha)$*, defining the integral transform* $\Psi_H\colon \mathtt{D}(Y)\to\mathtt{D}(X)$*, and an isomorphism* $\Psi_H \simeq \Phi_K^{-1}$*.*

For all $s \in \mathrm{S}$, assume the fibres $\mathrm{X}_s, \mathrm{Y}_s$ to be projective and connected. Finally, assume the fibre X_s to be Gorenstein with ample (or anti-ample) canonical bundle. ⋆

The definition of Bondal–Orlov point in the twisted case requires no additional effort.

Definition 8.5 – Assume to be working in Situation 8.4. Let $\operatorname{Spec} \mathrm{R} \to \mathrm{S}$ be of finite type. We say $\mathrm{P} \in \mathrm{D}(\mathrm{X}_\mathrm{R}, \alpha_\mathrm{R})$ is an *α-twisted Bondal–Orlov* point if the following hold.

(1) $\mathrm{P} \in \mathrm{D}^\mathrm{b}\mathrm{Coh}(\mathrm{X}_\mathrm{R}, \alpha_\mathrm{R})$.
(2) The natural map $\mathrm{R} \to \mathrm{Hom}(\mathrm{P}, \mathrm{P})$ is an isomorphism.
(3) For all $i < 0$, $\mathrm{Hom}(\mathrm{P}, \mathrm{P}[i]) = 0$.
(4) If R is a field, there exists $m \in \mathbf{Z}$ and an isomorphism $\Sigma \mathrm{P} \simeq \mathrm{P}[m]$.

We say such P is *universally Bondal–Orlov* if, for all $\mathrm{R} \to \mathrm{R}'$, the base change $\mathrm{P}_{\mathrm{R}'}$ is also a *Bondal–Orlov* point. ⋆

We write BO_α for the moduli of universal α-twisted Bondal–Orlov points. Just as in the untwisted case, BO_α comes with an action of both $\mathbf{Z}$ and $\mathbf{BG}_\mathrm{m}$. Clearly, the integral transform Φ_K induces an isomorphism of S-stacks $\mathrm{BO}_\alpha \simeq \mathrm{BO}_\beta$, compatible with the action $\mathbf{Z}$ and $\mathbf{BG}_\mathrm{m}$. Hence it descends to isomorphisms of S-stacks $\mathrm{BO}_\alpha/\mathbf{Z} \simeq \mathrm{BO}_\beta/\mathbf{Z}$ and $\mathrm{BO}_\alpha/\mathbf{Z}/\mathbf{BG}_\mathrm{m} \simeq \mathrm{BO}_\beta/\mathbf{Z}/\mathbf{BG}_\mathrm{m}$.

We have a twisted version of Lemma 5.5.

Lemma 8.6 – Assume to be working in Situation 8.4. There is an obvious map $\mathrm{Pt}_\alpha \to \mathrm{BO}_\alpha$ which is: fully faithful on objects, formally smooth and $\mathbf{BG}_\mathrm{m}$-equivariant. Same holds for the composition $\mathrm{Pt}_\alpha \to \mathrm{BO}_\alpha \to \mathrm{BO}_\alpha/\mathbf{Z}$. Finally, the induced morphism $\mathrm{Pt}_\alpha/\mathbf{BG}_\mathrm{m} \to \mathrm{BO}_\alpha/\mathbf{Z}/\mathbf{BG}_\mathrm{m}$ is a formally smooth monomorphism.

Ditto for β. ⋆

8.4. Bondal–Orlov. It will be no surprise that, because of the assumption on the canonical, we will show that $\mathrm{BO}_\alpha/\mathbf{Z} \simeq \mathrm{Pt}_\alpha$ where, using [11], the latter is the $\mathbf{G}_\mathrm{m}$-gerbe α itself. It also follows that $\mathrm{BO}_\alpha/\mathbf{Z}/\mathbf{BG}_\mathrm{m}$ is naturally isomorphic to X.

Proposition 8.7 – Assume to be working in Situation 8.4. Let $\operatorname{Spec} \mathrm{R} \to \mathrm{S}$ be of finite type with $\operatorname{Spec} \mathrm{R}$ connected. Let $\mathrm{E} \in \mathrm{BO}_\alpha(\mathrm{R})$. Then there exists (unique) r and $\mathrm{P} \in \mathrm{Pt}_\alpha(\mathrm{R})$ such that $\mathrm{E} \simeq \mathrm{P}[r]$. ⋆

Proof. This is the twisted version of Proposition 6.3. The argument is the same, as we can rely on Lemma 8.3 and Lemma 8.1. ∎

Proposition 8.8 – Assume to be working in Situation 8.4. The composition $\mathrm{Pt}_\alpha \to \mathrm{BO}_\alpha \to \mathrm{BO}_\alpha/\mathbf{Z}$ is an isomorphism of S-stacks, compatible with the action of $\mathbf{BG}_\mathrm{m}$. Moreover, we have isomorphisms $\mathrm{X} \simeq \mathrm{Pt}_\alpha/\mathbf{BG}_\mathrm{m} \simeq \mathrm{BO}_\alpha/\mathbf{Z}/\mathbf{BG}_\mathrm{m}$. ⋆

Proof. Just as in Proposition 6.4, the first result follows by considering the action of the constant sheaf $\mathbf{Z}$. The second follows as, using [11], we know Pt_α is the $\mathbf{G}_\mathrm{m}$-gerbe representing α. ∎

Theorem 8.9 – Assume to be working in Situation 8.4. Then there exists an isomorphism $f\colon \mathrm{Y} \to \mathrm{X}$ of S-stacks, such that $f^*\alpha = \beta$. ⋆

Proof. We argue as in Section 6 and show that the composition $\mathrm{Pt}_\beta \to \mathrm{BO}_\beta/\mathbf{Z} \to \mathrm{BO}_\alpha/\mathbf{Z} \simeq \mathrm{Pt}_\alpha$ is an isomorphism. By $\mathbf{BG}_\mathrm{m}$-equivariance, this induces a map $\mathrm{Y} \to \mathrm{X}$. We wish to show it is an isomorphism. Let $\mathrm{T} \to \mathrm{S}$ be faithfully flat, with T affine. It suffices to show the base change $Y_T \to X_T$ is an isomorphism. But this follows just as in the proof of Theorem 6.2, using Lemma 8.6.

The fact that $Pt_\beta \to Pt_\alpha$ is an isomorphism now follows by $\mathbf{BG}_m$-equivariance and the Lemma below. ■

Lemma 8.10 – Assume to be working in Situation 8.4. Let $Z, W \to Y$ be $\mathbf{G}_m$-gerbes and let $Z \to W$ be a $\mathbf{BG}_m$-equivariant morphism. Then $Z \to W$ is an isomorphism. ⋆

Proof. Indeed, the statement boils down to (an appropriate categorification of) the following trivial fact: if G is a group, any G-equivariant function $G \to G$ must be a bijection. Rigorously, let $T \to S$ be fully faithful, with T affine. It suffices to show the base change $Z_T \to W_T$ is an isomorphism. Let $U \to Y_T$ be a cover, trivializing both Z_T and W_T. Once again, it suffices to show that the base changes of Z_T and W_T over U are isomorphic. In other words, we have reduced the question to the following.

Let U be affine and let $U \times \mathbf{BG}_m \to U \times \mathbf{BG}_m$ be a $\mathbf{BG}_m$-equivariant map. Then $U \times \mathbf{BG}_m \to U \times \mathbf{BG}_m$ is an isomorphism of U-stacks. ■

8.5. Autoequivalences. We conclude with a description of the group of autoequivalences of the derived category of twisted sheaves.

Theorem 8.11 – Assume to be working in Situation 8.4 and assume S to be connected. Let $\mathcal{X} \to X$ be the $\mathbf{G}_m$-gerbe representing α and let $\mathrm{Aut}^{\mathbf{BG}_m}(\mathcal{X})$ be the group of $\mathbf{BG}_m$-equivariant automorphisms of $\mathcal{X}$. Then $\mathrm{Aut}_S(\mathtt{D}(X, \alpha)) \simeq \mathbf{Z} \times \mathrm{Aut}^{\mathbf{BG}_m}(\mathcal{X})$. ⋆

Proof. We may parallel Section 7. By looking at the effect on points, there is a group homomorphism $\mathrm{Aut}_S(\mathtt{D}(X)) \to \mathrm{Aut}^{\mathbf{Z}\times\mathbf{BG}_m}(BO_\alpha) = \mathbf{Z} \times \mathrm{Aut}^{\mathbf{BG}_m}(Pt_\alpha) = \mathbf{Z} \times \mathrm{Aut}^{\mathbf{BG}_m}(\mathcal{X})$. Recall that the abelian category $\mathtt{QCoh}(\mathcal{X})$ decomposes as a product $\mathtt{QCoh}(\mathcal{X}) = \prod_{n \in \mathbf{Z}} \mathtt{QCoh}(X, \alpha^n)$. Any element of $\mathrm{Aut}^{\mathbf{BG}_m}(\mathcal{X})$, which acts via pullback, preserves this decomposition. From this we deduce that the group homomorphism above is surjective. By arguing with kernels, we see that it is also injective. ■

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References

[1] B. Antieau, *A reconstruction theorem for abelian categories of twisted sheaves*, J. Reine Angew. Math. **712** (2016) 175–188.

[2] M. R. Ballard, *Derived categories of sheaves on singular schemes with an application to reconstruction*, Adv. Math. **227** (2011), no. 2, 895–919.

[3] P. Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005) 149–168.

[4] D. Ben-Zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966.

[5] D. Ben-Zvi, D. Nadler, and A. Preygel, *Integral transforms for coherent sheaves* (2013).

[6] R. Bezrukavnikov, I. Mirković, and D. Rumynin, *Singular localization and intertwining functors for reductive Lie algebras in prime characteristic*, Nagoya Math. J. **184** (2006) 1–55.

[7] B. Bhatt, *Algebraization and Tannaka duality*, ArXiv e-prints (2014)
[8] B. Bhatt and D. Halpern-Leistner, *Tannaka duality revisited*, ArXiv e-prints (2015)
[9] A. Bondal and D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Math. **125** (2001), no. 3, 327–344.
[10] M. Brandenburg, *Rosenberg's Reconstruction Theorem (after Gabber)*, ArXiv e-prints (2013)
[11] J. Calabrese and M. Groechenig, *Moduli problems in abelian categories and the reconstruction theorem*, Algebr. Geom. **2** (2015), no. 1, 1–18.
[12] A. Canonaco and P. Stellari, *Fourier-Mukai functors: a survey*, in Derived categories in algebraic geometry, EMS Ser. Congr. Rep., 27–60, Eur. Math. Soc., Zürich (2012).
[13] ———, *Internal Homs via extensions of dg functors*, Adv. Math. **277** (2015) 100–123.
[14] A. J. de Jong et al., *Stacks Project*, `http://math.columbia.edu/algebraic_geometry/stacks-git`.
[15] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France **90** (1962) 323–448.
[16] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford (2006), ISBN 978-0-19-929686-6; 0-19-929686-3.
[17] G. Laumon and L. Moret-Bailly, Champs algébriques, Vol. 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*, Springer-Verlag, Berlin (2000), ISBN 3-540-65761-4.
[18] M. Lieblich, *Twisted sheaves and the period-index problem*, Compos. Math. **144** (2008), no. 1, 1–31.
[19] V. A. Lunts and D. O. Orlov, *Uniqueness of enhancement for triangulated categories*, J. Amer. Math. Soc. **23** (2010), no. 3, 853–908.
[20] J. Lurie, *Tannaka Duality for Geometric Stacks* (2005) `arXiv:math/0412266v2`.
[21] H. J. Martinez Navas, Fourier-Mukai transform for twisted sheaves, Ph.D. thesis, Uni Bonn (2010).
[22] S. Mukai, *Duality between* $\mathrm{D}(X)$ *and* $\mathrm{D}(\hat{X})$ *with its application to Picard sheaves*, Nagoya Math. J. **81** (1981) 153–175.
[23] D. O. Orlov, *Equivalences of derived categories and* K3 *surfaces*, J. Math. Sci. (New York) **84** (1997), no. 5, 1361–1381. Algebraic geometry, 7.
[24] ———, *Derived categories of coherent sheaves and equivalences between them*, Uspekhi Mat. Nauk **58** (2003), no. 3(351), 89–172.
[25] A. Perego, *A Gabriel theorem for coherent twisted sheaves*, Math. Z. **262** (2009), no. 3, 571–583.
[26] A. Rizzardo, *Adjoints to a Fourier–Mukai transform*. `arXiv:1505.01035`.
[27] A. Rosenberg, *Spectra of 'spaces' represented by abelian categories* (2004) MPIM Preprints, 2004-115, `http://www.mpim-bonn.mpg.de/preblob/2543`.
[28] R. Rouquier, *Dimensions of triangulated categories*, J. K-Theory **1** (2008), no. 2, 193–256.
[29] ———, *Derived categories and algebraic geometry*, in Triangulated categories, Vol. 375 of *London Math. Soc. Lecture Note Ser.*, 351–370, Cambridge Univ. Press, Cambridge (2010).
[30] D. Rydh, *Noetherian approximation of algebraic spaces and stacks*, J. Algebra **422** (2015) 105–147.
[31] C. Sancho de Salas and F. Sancho de Salas, *Reconstructing schemes for the derived category*, Proc. Edinb. Math. Soc. (2) **55** (2012), no. 3, 781–796.
[32] B. Toën, *The homotopy theory of dg-categories and derived Morita theory*, Invent. Math. **167** (2007), no. 3, 615–667.
[33] ———, *Derived Azumaya algebras and generators for twisted derived categories*, Inventiones mathematicae **189** (2012), no. 3, 581–652.

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