CHOPPING UP DERIVED CATEGORIES

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Abstract. — This is an expanded version of a talk given during the 2014 Seattle Algebraic Geometry Workshop: new connections for recent PhDs, organized by Max Lieblich and Martin Olsson.

Contents

Derived categories were introduced by Verdier [**Ver96**] to provide a good framework to deal with homological algebra. For at least the last couple of decades (and with mirror symmetry providing a lot of motivation), there has been much work focused on studying purely $\mathtt{D}(X)$, the derived category of coherent sheaves on an algebraic variety X. In this note we will take the point of view that $\mathtt{D}(X)$ is a “linearization” of sorts of X: an algebraic gadget containing a great deal of information about X. We will have a look at what are called semiorthogonal decompositions of $\mathtt{D}(X)$ and we will catch a brief glimpse of Kuznetsov's homological projective duality [**Kuz07**].

Literature. — There are many sources from which to learn the basics of the subject. Most initiated swear by Huybrechts's book [**Huy06**] and its spiritual appendix on stability conditions [**Huy14**]. For a quick overview, I could not recommend more these six lectures by Shinder [**Shi**] and the notes by Bodzenta–Logvinenko of a lecture series delivered by Kuznetsov [**log12**]. One would be remiss without mentioning the trilogy of ICM addresses [**BO02**, **Bri06**, **Kuz14**]. Finally, Thomas's articles provide illuminating insight [**Tho01**, **Tho16**].

Conventions. — We work over the complex numbers. In what follows, X will denote a smooth and projective variety.

1. Basics

There seems to be a standard spiel to motivate derived categories. Generally one starts with some variation of "geometry is hard, algebra is easy" which (despite not being true) provides a guiding principle. It is certainly an old and tested idea to study geometric objects using algebraic invariants (such as homotopy groups, homology or Hodge theory). Here we want to use derived categories.

Consider $\mathtt{Vect}(X)$, the whole *category* of (algebraic) vector bundles on X. We can view this as a big invariant of X. Sadly $\mathtt{Vect}(X)$ is, categorically speaking, not a good choice: one cannot always take the cokernel of a vector bundle map (in other words $\mathtt{Vect}(X)$ is not an *abelian category*). To fix this, one passes to the bigger category $\mathtt{Coh}(X)$ of coherent sheaves (which is abelian).

Recall that, for an affine variety $\operatorname{Spec} R$, $\mathtt{Vect}(\operatorname{Spec} R)$ is the category of (finitely generated) projective R-modules and $\mathtt{Coh}(\operatorname{Spec} R)$ is the category of all (finitely generated) R-modules. For this reason, it is no surprise that passing from $\mathtt{Vect}(X)$ to $\mathtt{Coh}(X)$ is a good idea.

However, there are two reasons why we prefer $\mathtt{D}(X)$ to $\mathtt{Coh}(X)$. Reason number one is an old theorem of Gabriel: X is isomorphic to Y if and only if the categories $\mathtt{Coh}(X)$ and $\mathtt{Coh}(Y)$ are equivalent (see the original **[Gab62]** and generalizations [**Ros04**, **CG15**, **Bra13**]). In other words, $\mathtt{Coh}(X)$ is too rich! Reason number two is flexibility: even when $\mathtt{D}(X)$ is just as rich as $\mathtt{Coh}(X)$, the former supports a theory of decompositions which $\mathtt{Coh}(X)$ simply lacks.

Thus, for a better invariant we look at the derived category. We will write $\mathtt{D}(X)$ for the *bounded and coherent* derived category of X. This means the objects of $\mathtt{D}(X)$ are bounded chain complexes of coherent sheaves, while the morphisms are obtained by formally inverting all quasi-isomorphisms. We will now highlight some features of $\mathtt{D}(X)$ but we will not go into any details (once again, **[Tho01]** is highly recommended).

There is an inclusion $\mathtt{Coh}(X) \subset \mathtt{D}(X)$, which takes a sheaf F and treats it as a chain complex with zeros everywhere except in degree zero. An object $E \in \mathtt{D}(X)$, being a chain complex, has cohomology $H^i(E) \in \mathtt{Coh}(X)$ for every $i \in \mathbf{Z}$. There is a functor $[1]$ on $\mathtt{D}(X)$ called *shift* which takes a chain complex and moves everything one place to the left.

One does not have kernels and cokernels in derived categories (they are not abelian categories). However, $\mathtt{D}(X)$ is an example of a *triangulated category*. A short exact sequence in $\mathtt{Coh}(X)$

$$0 \to A \to B \to C \to 0$$

gives rise to what is called an *exact triangle*

$$A \to B \to C \to A[1]$$

in $\mathrm{D}(X)$. Exact triangles are, by definition, a specified class of triples of morphisms in $\mathrm{D}(X)$. An exact triangle $A \to B \to C \to A[1]$ has an associated long exact sequence

$$\cdots \to H^{i-1}(C) \to H^i(A) \to H^i(B) \to H^i(C) \to H^{i+1}(A) \to \cdots$$

of cohomology sheaves.

There are derived versions of the standard functors for coherent sheaves: Rf_*, Lf^*, $\overset{L}{\otimes}$, $R\underline{\mathrm{Hom}}$. These are computed by taking a suitable resolution and then applying the ordinary functor.

Convention. — In what follows, all functors will be implicitly derived, so that for example we will omit the R from Rf_*.

For any two $A, B \in \mathrm{Coh}(X)$ we have $\mathrm{Hom}_{\mathrm{D}(X)}(A, B[i]) = \mathrm{Ext}^i_{\mathrm{Coh}(X)}(A, B)$. We also have a couple of extra properties (coming from smoothness and properness of X). The category $\mathrm{D}(X)$ is Ext-finite: for any two $A, B \in \mathrm{D}(X)$, the $\mathbf{C}$-vector space

$$\bigoplus_i \mathrm{Hom}(A, B[i])$$

is finite-dimensional. We have *Serre duality*: for any two A, B there is an isomorphism

$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, A \otimes \omega_X[n])^\vee$$

where $\omega_X = \bigwedge^n \Omega_X$ is the canonical bundle and $n = \dim X$.

We want to abstract this last notion: we call $S_X(-) = (-) \otimes \omega_X[n]$ the *Serre functor* of $D(X)$. It is the unique functor making the Serre duality isomorphisms possible. Whenever a triangulated category D admits an auto-equivalence S_{D}, such that $\mathrm{Hom}(E, F) = \mathrm{Hom}(F, S_{\mathrm{D}}(E))^\vee$, we call S_{D} the Serre functor of D.

One might even think of any Ext-finite triangulated category with a Serre functor as being a "non-commutative" variety (or rather, the derived category of sheaves on a hypothetical non-commutative variety). It is useful at times to keep this point view in mind.

1.1. Derived invariants. — Now that we have $\mathrm{D}(X)$ it is natural to ask the following:

What information does $\mathrm{D}(X)$ *contain?*

More concretely, if $\mathrm{D}(X) \simeq \mathrm{D}(Y)$ what can we say about X and Y? For example, it's easy to show that $\dim X$ must be the same as $\dim Y$. Differently put, dimension is a *derived invariant.*

A fundamental theorem of Orlov tells us what functors between $\mathrm{D}(X)$ and $\mathrm{D}(Y)$ look like. Given $K \in \mathrm{D}(X \times Y)$ we define the *integral transform with kernel* K to be the functor

$$\Phi_K(E) := q_* \left(K \otimes p^* E\right)$$

where $p: X \times Y \to X$ and $q: X \times Y \to Y$ are the two projections. Orlov's theorem says all functors are integral transforms. We say the transform Φ_K is *Fourier–Mukai* if it is an equivalence.

For example, the identity $\mathrm{D}(X) \to \mathrm{D}(X)$ is given by the kernel $\mathcal{O}_\Delta$, with $\Delta \subset X \times X$ the diagonal. More generally, given a morphism $f: X \to Y$, pushforward f_* is given by the kernel $\mathcal{O}_{\Gamma_f}$, where $\Gamma_f \subset X \times Y$ is the structure sheaf of the graph of f. Similarly, for $g: Y \to X$, g^* is given by $\mathcal{O}_{\Gamma_g}$.

Orlov's theorem allows us to deduce interesting results. For example, Kodaira dimension is a derived invariant. As an ungraded vector space, singular cohomology is also a derived invariant. More precisely, even and odd cohomology are separately preserved while the individual pieces may be scrambled under a derived equivalence. See for example [**Abu15**] for more open questions in this area.

2. Decompositions

There exists a derived counterpart to Gabriel's theorem. Let X and Y be two smooth and projective varieties and assume X has ample (or anti-ample) canonical bundle. If $\mathtt{D}(X) \simeq \mathtt{D}(Y)$ then $X \simeq Y$ (see [**BO01**] for the original article and [**Rou10**, **Bal11**, **SdSSdS12**, **MN10**, **Cal16**] for generalizations). As discussed earlier, even for varieties where $\mathtt{D}(X)$ recovers X, derived categories provide extra flexibility in the form of decompositions.

Suppose for a moment that $X = Y \amalg Z$ is disconnected. If E is some sheaf on X, then there is a natural morphism $E \to j_* j^* E$ which is surjective with kernel $i_* i^* E$, where i, j are the inclusions of respectively Y, Z. For E a complex in $\mathtt{D}(X)$ it's exactly the same, only we say there is an exact triangle

$$i_* i^* E \to E \to j_* j^* E \to i_* i^* E[1].$$

Of course, in this special setup a much stronger statement is true: since we may swap the roles of Y and Z, this exact triangle splits as a direct sum. Since this fact holds for all E, we should be able to say that the category $\mathtt{D}(X)$ itself splits as a sum of $\mathtt{D}(Y)$ and $\mathtt{D}(Y)$. Let us make this precise.

Definition 2.1. — Let $\mathtt{A}, \mathtt{B}$ be two triangulated subcategories of $\mathtt{D}(X)$. We say that $\mathtt{A}, \mathtt{B}$ form an *orthogonal decomposition* (OD) of $\mathtt{D}(X)$ if

- $\mathrm{Hom}(\mathtt{B}, \mathtt{A}) = 0 = \mathrm{Hom}(\mathtt{A}, \mathtt{B})$ "orthogonality"
- for any E there is a unique exact triangle

$$A \to E \to B \to A[1]$$

  with $A \in \mathtt{A}$, $B \in \mathtt{B}$.

Sometimes we refer to the second condition as expressing the "fullness" of the decomposition. The following result is then no surprise.

Proposition 2.2. — X is disconnected if and only if it admits a (non-trivial) orthogonal decomposition.

So we learned that $\mathtt{D}(X)$ knows when X is connected. Now, while there isn't a topological intermediate between being connected and being disconnected, on the algebraic side one can weaken the definition.

Definition 2.3. — A pair of triangulated subcategories $\mathtt{A}, \mathtt{B} \subset D(X)$ form a *semiorthogonal decomposition* (ØD) of $\mathtt{D}(X)$ if

- $\mathrm{Hom}(\mathtt{A}, \mathtt{B}) = 0$

– for any E there is a unique exact triangle

$$A \to E \to B \to A[1]$$

with $A \in \mathtt{A}$, $B \in \mathtt{B}$.

When this occurs we write $\mathtt{D}(X) = \langle \mathtt{B}, \mathtt{A} \rangle$.

Of course, the definition did not use anything special about $\mathtt{D}(X)$ and makes sense in any triangulated category. For example, one might be able to decompose $\mathtt{A}$ or $\mathtt{B}$ further. In general, a ØD of $\mathtt{D}(X)$ consists of a finite number of pieces and one writes $\mathtt{D}(X) = \langle \mathtt{A}_1, \dots, \mathtt{A}_n \rangle$. Being able to do things like this is one of the reasons we love algebra.

Before giving the first example let's make a silly observation. Write $\mathtt{D}(\mathbf{C})$ for the derived category of $\mathbf{C}$-vector spaces. Notice that $\mathtt{D}(\mathbf{C}) = \mathtt{D}(\mathrm{pt})$, the derived category of a point. Let $E \in D(X)$. There is a functor $\phi_E \colon \mathtt{D}(\mathbf{C}) \to \mathtt{D}(X)$ which takes a complex of vector spaces V to $V \otimes_{\mathbf{C}} E$. The image of ϕ_E is the smallest triangulated subcategory of $\mathtt{D}(X)$ containing E. However, this category need not be the same as $\mathtt{D}(\mathbf{C})$, as ϕ_E may not be fully faithful.

Definition 2.4. — An object E is called *exceptional* if ϕ_E is fully faithful. More concretely, E is exceptional if and only if $\mathrm{Hom}(E, E) = \mathbf{C}$ and $\mathrm{Hom}(E, E[k]) = 0$ for $k \neq 0$.

By the way, a semiorthogonal decomposition consisting purely of exceptional objects is called a *full exceptional collection.* That's a great thing to have.

***Theorem 2.5* (Beilinson).** — Projective spaces admit a full exceptional collection.

$$\mathtt{D}(\mathbf{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle$$

where (to be pedantic) one should have written $\langle \phi_{\mathcal{O}}(\mathrm{D}(\mathbf{C})), \dots, \phi_{\mathcal{O}(n)}(\mathrm{D}(\mathbf{C})) \rangle$.

Caution. — Caution: the notation for ØDs is very misleading, in the sense that it lacks crucial information. Consider the following three categories. The first two are $\mathtt{T}_1 = \mathtt{D}(\mathrm{pt} \sqcup \mathrm{pt})$, $\mathtt{T}_2 = \mathtt{D}(\mathbf{P}^1)$. For the third, consider $\mathtt{D}(\mathbf{P}^2) = \langle \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \rangle$ and take $\mathtt{T}_3 = \langle \mathcal{O}, \mathcal{O}(1) \rangle$, the smallest triangulated category containing $\mathcal{O}$ and $\mathcal{O}(1)$. In all three cases we have $\mathtt{T}_i = \langle \mathtt{D}(\mathbf{C}), \mathtt{D}(\mathbf{C}) \rangle$, however the three categories are very different.

The decomposition in $\mathtt{T}_1$ is completely orthogonal, so there are no maps between the two components in either direction. For $\mathtt{T}_2$, the decomposition is not orthogonal as $\mathrm{Hom}_{\mathbf{P}^1}(\mathcal{O}, \mathcal{O}(1)) = \mathbf{C}^2$. The third case is similar, however $\mathtt{T}_3 \neq \mathtt{T}_2$. Indeed, $\mathrm{Hom}_{\mathbf{P}^2}(\mathcal{O}, \mathcal{O}(1)) = \mathbf{C}^3$. Since one can easily classify exceptional collections on $\mathbf{P}^1$, one concludes that $\mathtt{T}_3$ cannot be $\mathtt{D}(\mathbf{P}^1)$.

What the notation is hiding is the data of the maps going in opposite direction, which can be thought as some gluing data (the intuition comes from topology, specifically the theory of perverse sheaves **[Wil]**).

Non-example. — Assume X is K-trivial, i.e. $\omega_X \cong \mathcal{O}_X$. When this holds, the Serre functor of $D(X)$ is just the shift $[d]$ where $d = \dim X$. In this case we say $\mathtt{D}(X)$ is a *Calabi–Yau* of dimension d. Calabi–Yau categories admit only the trivial semiorthogonal decomposition.

2.1. Examples. — We list some standard techniques to produce decompositions.

***Theorem* (Orlov [Orl92]).** — Let E be a vector bundle of rank $n+1$ on X and let $p\colon \mathbf{P}_{\mathrm{X}}(\mathrm{E}) \to \mathrm{X}$ the projectivization. Let $\mathcal{O}_p(1)$ be the relative tautological line bundle. Then

$$\mathtt{D}(\mathbf{P}(\mathrm{E})) = \langle p^*\mathtt{D}(\mathrm{X}), p^*\mathtt{D}(\mathrm{X}) \otimes \mathcal{O}_p(1), \ldots, p^*\mathtt{D}(\mathrm{X}) \otimes \mathcal{O}_p(n)\rangle.$$

There is also a twisted version.

***Theorem* (Bernardara [Ber09]).** — Let $p\colon \mathrm{Y} \to \mathrm{X}$ a Brauer–Severi variety and let β be the corresponding Brauer class.

$$\mathtt{D}(\mathrm{Y}) = \langle p^*\mathtt{D}(\mathrm{X}), p^*\mathtt{D}(\mathrm{X}, \beta^{-1}) \otimes \mathcal{O}_p(1), \ldots, p^*\mathtt{D}(\mathrm{X}, \beta^{-n}) \otimes \mathcal{O}_p(n)\rangle.$$

Recall that a Brauer–Severi variety is a an étale $\mathbf{P}^n$-bundle. Concretely, there exists an étale cover of X, such that on each patch $\mathrm{U} \subset \mathrm{X}$, one has $p^{-1}(\mathrm{U}) = \mathrm{U} \times \mathbf{P}^n$. These gadgets are PGL_{n+1}-torsors and are thus classified by $\mathrm{H}^1_{\text{ét}}(\mathrm{X}, \mathrm{PGL}_{n+1})$. There is a short exact sequence $0 \to \mathbf{G}_{\mathrm{m}} \to \mathrm{GL}_{n+1} \to \mathrm{PGL}_{n+1} \to 1$. So from a Brauer-Severi variety we get a class $\beta \in \mathrm{H}^2_{\text{ét}}(\mathrm{X}, \mathrm{G}_{\mathrm{m}})$, which is called the ***Brauer class*** of p. One can use β to define β-***twisted sheaves*** which are assembled in an abelian category $\mathtt{Coh}(\mathrm{X}, \beta)$. By $\mathtt{D}(\mathrm{X}, \beta)$ we mean its derived category. Finally, it should be pointed out that $\mathcal{O}_p(-i)$ is actually well-defined only as an $f^*\beta^i$-twisted sheaf.

Finally, we have the ***blowup formula***.

***Theorem* (Orlov [Orl92]).** — Let $\mathrm{Y} \subset \mathrm{X}$ be a smooth subvariety of codimension c. Blow it up.

$$\begin{tikzcd} \mathrm{E} \arrow[r, "i"] \arrow[d, "p"'] & \mathrm{Bl}_{\mathrm{Y}}\,\mathrm{X} \arrow[d, "\pi"] \\ \mathrm{Y} \arrow[r] & \mathrm{X} \end{tikzcd}$$

Because we are assuming smoothness, p is actually the projectivization of the normal bundle of $\mathrm{Y} \subset \mathrm{X}$. We have the following decomposition.

$$\mathtt{D}(\mathrm{Bl}_{\mathrm{Y}}\,\mathrm{X}) = \langle \pi^*\mathtt{D}(\mathrm{X}), i_*p^*\mathtt{D}(\mathrm{Y}), i_*p^*\mathtt{D}(\mathrm{Y}) \otimes \mathcal{O}_p(1), \ldots, i_*p^*\mathtt{D}(\mathrm{Y}) \otimes \mathcal{O}_p(c-2)\rangle$$

2.2. Mutations. — Another natural question in the theory is how many decompositions does $\mathtt{D}(\mathrm{X})$ admit, if any? Apart from few special cases (such as the Calabi–Yau case or curves) this question seems hard to answer. On the other hand, given a decomposition there are operations called ***mutations*** which one can perform. Assume we have a semiorthogonal decomposition $\mathtt{D}(\mathrm{X}) = \langle \mathtt{A}_1, \ldots, \mathtt{A}_n\rangle$ and fix i. We have two functors (which we will not define): $\mathrm{L}_{\mathtt{A}_i}, \mathrm{R}_{\mathtt{A}_i}\colon \mathtt{D}(\mathrm{X}) \to \mathtt{D}(\mathrm{X})$, the left and right mutations along $\mathtt{A}_i$. One has that $\mathrm{L}_{\mathtt{A}_i}$ is fully faithful when restricted to $\mathtt{A}_{i+1}$ and $\mathrm{R}_{\mathtt{A}_i}$ is fully faithful when restricted to $\mathtt{A}_{i-1}$. However, $\mathrm{L}_{\mathtt{A}_i}(\mathtt{A}_i) = 0 = \mathrm{R}_{\mathtt{A}_i}(\mathtt{A}_i)$. But most importantly, we have two new decompositions

$$\mathtt{D}(\mathrm{X}) = \langle \mathtt{A}_1, \ldots, \mathtt{A}_{i-1}, \mathrm{L}_{\mathtt{A}_i}\mathtt{A}_{i+1}, \mathtt{A}_i, \mathtt{A}_{i+2}, \ldots, \mathtt{A}_n\rangle$$
$$\mathtt{D}(\mathrm{X}) = \langle \mathtt{A}_1, \ldots, \mathtt{A}_{i-2}, \mathtt{A}_i, \mathrm{R}_{\mathtt{A}_i}\mathtt{A}_{i-1}, \mathtt{A}_{i+1}, \ldots, \mathtt{A}_n\rangle.$$

Because of fully faithfulness, the categories $\mathtt{A}_{i+1}$ and $\mathrm{L}_{\mathtt{A}_i}\mathtt{A}_{i+1}$ are abstractly isomorphic. However, the way they are embedded in $\mathtt{D}(\mathrm{X})$ changes quite a bit. This is similar Beilinson's decomposition: all the components $\langle \mathcal{O}(i) \rangle$ are equivalent to $\mathsf{D}(\mathrm{pt})$, but they are embedded in $\mathsf{D}(\mathbf{P}^n)$ in very different ways. In the case of a decomposition given by an exceptional collection, the formulas for mutations become a great deal more explicit. The composition of mutations also has an interesting structure, one sees for example an action of the braid group. For more on this, see [**Shi**, Lecture 2].

2.3. Phantoms. — Here is another important question: given a two subcategories satisfying the first axiom of being a semiorthogonal decomposition, is there an efficient criterion to tell if they also satisfy axiom two? Put differently, how do we tell whether a semiorthogonal collection of subcategories is full (i.e. it spans the whole derived category)? Even for exceptional collections there seems to be no good answer in general.

Recall that the *Grothendieck group* $\mathrm{K}_0(\mathrm{X})$ is the free abelian group generated by isomorphism classes $[\mathrm{E}]$ of complexes $\mathrm{E} \in \mathsf{D}(\mathrm{X})$ subject to the relations $[\mathrm{A}] = [\mathrm{B}] + [\mathrm{C}]$ whenever there is an exact triangle $\mathrm{A} \to \mathrm{B} \to \mathrm{C} \to \mathrm{A}[1]$.

Remark. — Whenever you have a semiorthogonal decomposition the Grothendieck group splits. Precisely, if $\mathsf{D}(\mathrm{X}) = \langle \mathsf{A}_1, \ldots, \mathsf{A}_n \rangle$, then $\mathrm{K}_0(\mathrm{X}) = \bigoplus_i \mathrm{K}_0(\mathsf{A}_i)$.

It was conjectured for a while that if you had a semiorthogonal collection, $\mathsf{A}_1, \ldots, \mathsf{A}_n$, such that $\bigoplus_i \mathrm{K}_0(\mathsf{A}_i) = \mathrm{K}_0(\mathrm{X})$ then the collection had to be full. Any collection can be completed to a decomposition, just by considering the subcategory orthogonal to the rest. In other words, given a semiorthogonal collection $\mathsf{A}_1, \ldots, \mathsf{A}_r$, we always have

$$\mathsf{D}(\mathrm{X}) = \langle \mathsf{A}_1, \ldots, \mathsf{A}_r, \mathsf{B} \rangle$$

where B consists of all objects B such that $\mathrm{Hom}(\mathrm{B}, \mathrm{A})$ for all $\mathrm{A} \in \mathsf{A}_i$ and for all i. So the question boils down to the following: can one find an example of a B such that $\mathrm{K}_0(\mathsf{B}) = 0$?

Such categories are called *phantoms* and, sadly, they do exist (even inside surfaces!). See [**GO13**, **BGvBKS15**, **Kuz15**] and also [**Sos15**] for an interesting list of open questions.

3. Homological Projective Duality

We devote the rest of this article to a glimpse of Kuznetsov's *homological projective duality* (HPD). Consider a hypersurface $\mathrm{X} \subset \mathbf{P}^{\mathrm{N}}$ of degree $d \leq \mathrm{N}$. We have a decomposition

$$\mathsf{D}(\mathrm{X}) = \langle \mathsf{A}_{\mathrm{X}}, \mathcal{O}_{\mathrm{X}}, \mathcal{O}_{\mathrm{X}}(1), \ldots, \mathcal{O}_{\mathrm{X}}(\mathrm{N} - d) \rangle$$

where A_{X} is (by definition) the orthogonal of the category spanned by $\mathcal{O}_{\mathrm{X}}, \ldots, \mathcal{O}_{\mathrm{X}}(\mathrm{N} - d)$ [**Kuz06**]. When X is understood, we drop the subscript and simply write A. Notice that the Bondal–Orlov theorem applies here, given that the canonical of X is anti-ample.

In some cases, the "interesting" component A seems to be a rich invariant of X. For example, we have the following result.

Theorem 3.1 **(Bernardara-Macrì-Mehrotra [BMMS12]).** — Let X and Y be two cubic threefolds with respective components A_{X}, A_{Y}. If $\mathsf{A}_{\mathrm{X}} \simeq \mathsf{A}_{\mathrm{Y}}$ then $\mathrm{X} \simeq \mathrm{Y}$.

A natural question is if there are any general properties satisfied by the category A. Being a component of $\mathsf{D}(X)$, A is Ext-finite and admits a Serre functor S_{A}. While it is not easy to give an explicit formula for S_{A}, Kuznetsov deduced the following striking property.

$$S_{\mathsf{A}}^{d/\gcd(N+1,d)} = \left[\frac{(d-2)(N+1)}{\gcd(N+1,d)}\right].$$

In other words, a power of the Serre functor is always a shift! Phrased differently, A is a Calabi–Yau category of *fractional* dimension. See [**Kuz16**] for a general discussion of these categories.

If one chooses N and d carefully, one can arrange for A to be genuinely Calabi-Yau. This occurs in the case of cubic fourfolds. If $X \subset \mathbf{P}^5$ is a cubic, the corresponding component A is Calabi-Yau of dimension 2. Since two-dimensional Calabi-Yau varieties are the K3 surfaces, a natural question arises:

is there a K3 surface Y such that $\mathsf{D}(Y) \simeq \mathsf{A}$?

It is easy to see that generically (in the space of cubic fourfolds) this cannot be true: the Grothendieck group of A is simply of the wrong rank. However, in [**Kuz10**], Kuznetsov studies some well known examples of rational cubic fourfolds and formulates the following conjecture.

Conjecture 3.2 **(Kuznetsov).** — Let X be a cubic fourfold and let A_X be the corresponding component of $\mathsf{D}(X)$. Then there exists a K3 surface Y and an equivalence $Y \simeq \mathsf{A}_X$ if and only if X is rational.

If true, this conjecture would easily imply that the generic cubic fourfold is irrational. The magnitude of such a fact should be highly emphasized. It has been expected for a long time that the generic cubic fourfold is irrational and much research is being devoted to finding a proof. It should be noted that Kuznetsov's conjecture does not come from a vacuum: Hassett had already noticed a relationship between the Hodge theory of cubic fourfolds and related K3 surfaces [**Has00**]. See the remarkable paper [**AT14**] for the relationship between the derived category and the Hodge theoretic points of view. See also [**Huy15**] where the properties of the A_X category are explored further. We must also mention that (in spite of expectations of their scarcity) *new* examples of rational cubic fourfolds have been found [**AHTVA16**]. Finally, Huybrechts and Rennemo have recently announced a proof of Torelli for cubic fourfolds which goes through the A_X category.

3.1. Families. — Let us go back to the general theory. Consider again projective space P^N and fix $d \leq N$. So far we've had a look at a fixed hypersurface $X \subset \mathbf{P}^N$. The obvious next step is to see what happens in a family. Consider two hypersurfaces X_0, X_1. We can do two things: intersect them or consider the pencil spanned by them. We assume that the former $Z = X_0 \cap X_1$ is smooth of dimension $N-2$. If f_0, f_1 are the equations defining X_0 and X_1, the total space H of the pencil is described by the equation $y_0 f_0 + y_1 f_1 = 0$ in $\mathbf{P}^N \times \mathbf{P}^1$, where $[y_0, y_1]$ are coordinates on $\mathbf{P}^1$. Drawing a diagram might be useful.

$$\begin{array}{ccc} H_t & \longrightarrow & H \\ \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{t} & \mathbf{P}^1 \end{array}$$

Given a point $t \in \mathbf{P}^1$, above we wrote H_t for the fibre of H at t. Concretely, if $t = [a, b] \in \mathbf{P}^1$, H_t is the hypersurface cut out by the equation $af_0 + bf_1 = 0$ in $\mathbf{P}^N$. We have $H_0 = X_0$ and $H_\infty = X_1$.

Kuznetsov further observed not only the existence of a decomposition

$$D(Z) = \langle A_Z, \mathcal{O}_Z, \dots, \mathcal{O}_Z(N - 2d)\rangle$$

(we are implicitly assuming $2d \le N$) but also of a decomposition

$$D(H) = \langle A_H, \mathcal{O}, \mathcal{O}(0,1), \mathcal{O}(1,0), \mathcal{O}(1,1), \dots, \mathcal{O}(d,0), \mathcal{O}(d,1)\rangle.$$

For each (smooth) cubic fourfold X_t in the pencil, we have a corresponding component A_{H_t}. The idea is that A_H is the "total space" of all these categories. Indeed, Kuznetsov shows that the image of A_H (via pullback along $H_t \to H$) inside $D(X_t)$ is precisely A_{X_t}! But what role does Z play? It turns out that A_Z is equivalent to A_H. This baby case of HPD can actually be proved directly: since H is the blowup of $\mathbf{P}^N$ along Z, Orlov's formula (plus a few mutations) does the job (see [**Tho16**] for details, or [**CT16**] for the case of cubic fourfolds).

But more is true. This relationship between derived categories of total spaces of linear systems and base loci is true more generally. Remarkably, there is also a basechange compatibility when passing to different subsystems and results also hold for singular varieties. HPD is a very robust theory. But the true testament of its strength is the wealth of examples and of results which it has inspired [**Del11**, **CT16**, **BBF16**, **BT16**, **BDF$^+$13**, **BDF$^+$14**, **CT15**, **Tab16**, **Vol15**, **HT13**, **HT15**, **Per**].

What we have not explained yet is the word ***duality*** in HPD. Indeed, the whole point of the theory is that these A categories should have a geometric counterpart. We will explain this for quadrics in the next section.

4. Quadrics

Quadrics are degree two hypersurfaces and we will stick to the even dimensional case for simplicity. Kapranov showed that for a smooth quadric $Q \subset \mathbf{P}^{2N-1}$ there is a decomposition

$$D(Q) = \langle \mathcal{O}(-2n + 3), \dots, \mathcal{O}(-1), \mathcal{O}, S_+, S_-\rangle$$

where $S_\pm$ are the so-called ***spinor vector bundles.*** For a quadric surface $Q \simeq \mathbf{P}^1 \times \mathbf{P}^1$ they can be identified with $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$. As objects of $D(Q)$, S_+, S_- are completely orthogonal. In other words $\langle S_+, S_-\rangle = D(\mathrm{pt} \sqcup \mathrm{pt})$. Thus, up to mutations, we have described Kuznetsov's category $A_Q = \langle S_+, S_-\rangle = D(\mathrm{pt} \sqcup \mathrm{pt})$.

Suppose now we have a family of quadrics $H \to L$, over a base variety L. For each $l \in L$ we write H_l for the fibre. We are interested in the component A_l, spanned by the two spinor bundles on H_l. As l moves in L, the two spinor bundles vary. Intuitively, we expect $D(H)$ to have a decomposition with an interesting component A_H, which serves as the total space for all the categories A_l. We then would like to consider the double cover $Y \to L$, whose fibres Y_l correspond to the two spinor bundles. Finally, we expect the "universal spinor bundles" on $Q \times_L Y$ to be the kernel of a Fourier–Mukai equivalence $\mathtt{D}(Y) \simeq \mathtt{A}_Q$.

There are indeed cases where this geometric description can be made rigorous, with a few hiccups. One issue is that when $\dim L \geq 2$ a universal spinor bundle needn't exist. In moving in a family there is an obstruction to keeping track of the order of S_+, S_- and one has not a universal sheaf, but rather a *twisted* sheaf. In other words, there is a Brauer class $\alpha \in H^2_{\text{ét}}(Y, \mathbb{G}_m)$ encoding how much S_+, S_- are shuffled around. We would then expect $\mathtt{D}(Y, \alpha)$ – the derived category of α-twisted sheaves – to be equivalent to $\mathtt{A}_Q$. A second issue is that singularities might make life impossible.

Let us now revert to the more rigid setup of a linear family $L \subset \mathbf{P}H^0(\mathbf{P}^{2N-1}, \mathcal{O}(2))$ of quadrics. Let $H \subset L \times \mathbf{P}^{2N-1}$ be the total space. For each $l \in L$ we have the fibre H_l and corresponding component $\mathtt{A}_l$. Let us draw a diagram.

$$\begin{tikzcd} H_l \arrow[r] \arrow[d] & H \arrow[r, hook] \arrow[d] & L \times \mathbf{P}^{2N-1} \\ \mathrm{pt} \arrow[r, "l"] & L & \end{tikzcd}$$

In this setup we actually know there is a decomposition of $\mathtt{D}(H)$ with the interesting component $\mathtt{A}_H$ one side and line bundles of the form $\mathcal{O}(i, j)$ on the other. As before, the image of $\mathtt{A}_H$ under pullback along $H_l \to H$ is $\mathtt{A}_l$.

Consider the case where $\dim L = 1$, a pencil of quadrics. Assume L is spanned by Q_0, Q_1 and that $Q_0 \cap Q_1$ is smooth of the expected dimension. Let $Y \to L$ be the double cover of L branched over the locus of singular quadrics. In this case, Y is indeed a fine moduli space of spinor bundles for $H \to L$. Bondal and Orlov proved [**BO95**] that (generically) the following decomposition holds

$$\mathtt{D}(Q_0 \cap Q_1) = \langle \mathcal{O}(2N-5, \ldots, \mathcal{O}(-1), \mathcal{O}, \mathtt{D}(Y) \rangle.$$

So here we already see both features of HPD: using derived categories we relate the base locus $Q_0 \cap Q_1$, to the total space H and to the "dual" Y.

Let's move on to $\dim L = 2$. Assume again L to be spanned by Q_0, Q_1, Q_2 and that $Q_0 \cap Q_1 \cap Q_2$ is smooth and of the expected dimension. Again, we can consider the double cover $Y \to L$ branched over the singular quadrics. This time, Y is no longer *fine.* As mentioned earlier, there is an obstruction in the form of a Brauer class $\alpha \in H^2_{\text{ét}}(Y, \mathbf{G}_m)$. Nevertheless, Căldăraru [**Cal00**] was able to show that the following decomposition holds.

$$\mathtt{D}(Q_0 \cap Q_1 \cap Q_2) = \langle \mathcal{O}(2N-7), \ldots, \mathcal{O}(-1), \mathcal{O}, \mathtt{D}(Y, \alpha) \rangle.$$

Finally, let us move to $\dim L = 3$. Here things get even more problematic: the double cover Y is no longer smooth. Addington solved this problem by introducing a well-behaved notion of spinor sheaf for the singular quadrics [**Add11**]. After taking a suitable (non-Kähler!) resolution $\tilde{Y} \to Y$, one still has a decomposition [**Add09**]

$$\mathtt{D}(Q_0 \cap Q_1 \cap Q_2 \cap Q_3) = \langle \mathcal{O}(2N+9), \ldots, \mathcal{O}(-1), \mathcal{O}, \mathtt{D}(\tilde{Y}, \alpha) \rangle.$$

However, for $\dim L = 4$ and above I do not know of any successful geometric approach to this problem.

4.1. Clifford algebras. — Homological projective duality gives an alternative (more algebraic) explanation of the equivalences discussed above, which however works in any dimension. Recall that, given a quadratic form q on a vector space V, the *Clifford algebra* of q is the quotient of the tensor algebra of V by the ideal generated by $v \otimes v - q(v)$ for $v \in \mathrm{V}$. Let us write $\mathrm{C} = \mathrm{C}(\mathrm{V}, q)$ for this algebra. It is $\mathbb{Z}/2\mathbb{Z}$-graded. We write C_0 for the even-degree subalgebra. The category of C_0-modules is actually easy to describe: the algebra C_0 splits as a product of two matrix algebras (because we are working over the complex numbers and with even-dimensional quadrics). By Morita theory, we then know that $\mathtt{D}(\mathrm{C}_0) = \mathtt{D}(\mathbf{C} \times \mathbf{C}) = \mathtt{D}(\mathrm{pt} \sqcup \mathrm{pt})$. This should remind us of the component $\mathtt{A}_\mathrm{Q}$ of the corresponding quadric!

First off, it is easy to make sense of families of Clifford algebras on a variety (by replacing vector spaces with sheaves). If $\mathrm{L} \subset \mathbf{P}\mathrm{H}^0(\mathbf{P}^{2\mathrm{N}-1}, \mathcal{O}(2))$ is a linear space of qudarics, we have a sheaf C_0 of (even-degree) Clifford algebras on L. Over $l \in \mathrm{L}$, the fibre of C_0 will be the Clifford algebra of the quadratic form corresponding to the quadric H_l. Let us write $\mathtt{D}(\mathrm{L}, \mathrm{C}_0)$ for the derived category of coherent sheaves on L equipped with the structure of a module over C_0.

Assume L is spanned by quadrics $\mathrm{Q}_0 \cap \cdots \cap \mathrm{Q}_{m-1}$ such that $m \leq \mathrm{N}$ and their intersection Z is smooth of the expected dimension. Then Kuznetsov showed [**Kuz08**] that $\mathtt{D}(\mathrm{Z})$ has a decomposition

$$\langle \mathcal{O}(-2\mathrm{N} + m + 1), \ldots, \mathcal{O}(-1), \mathcal{O}, \mathtt{D}(\mathrm{L}, \mathrm{C}_0) \rangle.$$

One can more or less directly recover the equivalences above. Indeed, the algebra C_0 has a center A. The relative spectrum of A defines a variety $\mathrm{Y} \to \mathrm{L}$, which is a double cover branched over the locus of singular quadrics. For $\dim \mathrm{L} = 2$, the algebra C_0 (when viewed as defined on Y) turns out to be an Azumaya algebra. An Azumaya gives rise to a Brauer class α: the category of C_0-modules is equivalent to that of α-twisted sheaves over Y.

When the dimension of L is higher, the nature of the algebra C_0 is more complicated. Nevertheless, this dual (non-commutative) variety $(\mathrm{L}, \mathrm{C}_0)$ gives a powerful description of the derived category of a (complete) intersection of quadrics.

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April 24, 2017

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