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A note on derived equivalences and birational geometry

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Abstract

We give a straightforward proof of the following fact. Let X and Y be two smooth (or Gorenstein) and projective varieties and let $\Phi\colon D(X) \hookrightarrow D(Y)$ be a fully faithful functor between their derived categories. If Φ sends a skyscraper to a skyscraper, then X and Y are birational. We also show some related results.

1. *Introduction*

After the seminal paper of Mukai [**17**], there has been a surge of research devoted to understanding derived categories of coherent sheaves. In the 1990s, Bondal and Orlov found a strong link with birational geometry [**4**] (see also [**9**]). Namely, they showed that when X and Y are related by a *standard flip* there is a fully faithful embedding $D(Y) \hookrightarrow D(X)$. In the case of a *standard flop* they showed that this embedding is an equivalence $D(Y) \simeq D(X)$.

Flips play a crucial role in the minimal model program and, by a result of Kawamata [**12**], any two birational minimal models are connected by a chain of flops (in particular, this applies to all birational Calabi-Yau varieties). Given all this evidence, one might even dream to detect minimal models of a variety X as special subcategories sitting inside $D(X)$. Though this dream is very far from reality, Bondal and Orlov conjectured that if two (smooth and projective) varieties are related by a flop then their derived categories should be equivalent.

This conjecture is a theorem of Bridgeland [**5**] in dimension 3, reinterpreted noncommutatively by van den Bergh [**21**] (see also [**1**, **7**]).

In this paper, we are concerned with the reverse question. Assume X and Y are two smooth and projective varieties with equivalent derived categories $D(X) \simeq D(Y)$. Already in dimension two there exist examples with X and Y nonbirational. On the other hand, if the canonical bundle of X is ample (or anti-ample), Y is actually isomorphic to X. If we know $\mathrm{kod}(X) = \dim X$, then X and Y are birational (they are in fact K-equivalent).

In the present work, instead of restricting the types of varieties, we strengthen the functors. Namely, we prove the following criterion.

Theorem 1.1. *Let X and Y be two geometrically integral, Gorenstein and projective varieties over a field $\mathtt{k}$ and let $\Phi\colon D(X) \to D(Y)$ be a fully faithful functor between their derived categories (see Section 3.1 for details). Assume there are simple sheaves $E \in \mathcal{G}_X(k)$, $F \in \mathcal{G}_Y(k)$ and an isomorphism $\Phi_k(E) \simeq F$, then $\mathcal{G}_X$ and $\mathcal{G}_Y$ are birational.*

Here $\mathcal{G}_X$ denotes any open irreducible subset of the moduli space of simple coherent sheaves on X (and analogous notation for $\mathcal{G}_Y$). Finally, Φ_k is the scalar extension of Φ to $D(X_k) \to D(Y_k)$. From this, what was promised in the abstract follows easily.

Corollary 1.2. *With assumptions as above, suppose there are closed k-points $x \in X(k)$, $y \in Y(k)$ and an isomorphism $\Phi_k(\kappa(x)) \simeq \kappa(y)$. Then X and Y are birational.*

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Here $\kappa(z)$ is the skyscraper at the closed point z with stalk the residue field of z.

Corollary 1.3. *With the assumptions as above, suppose there exist line bundles $L \in \mathrm{Pic}_X(k)$, $L' \in \mathrm{Pic}_Y(k)$ and an isomorphism $\Phi_k(L) \simeq L'$. Then Pic_X^L and $\mathrm{Pic}_Y^{L'}$ are birational. Moreover, if Pic_X^0 and Pic_Y^0 are abelian varieties, then Pic_X^L and $\mathrm{Pic}_Y^{L'}$ are isomorphic.*

Here we wrote Pic_X^L for the component of Pic_X containing L.

Strategy

The idea of the proof is quite simple. Consider $\mathcal{M}_X$, the moduli space of objects of $D(X)$ and the embedding $X \hookrightarrow \mathcal{M}_X$, which sends a point $x \in X$ to the corresponding skyscraper $\kappa(x)$. Similarly, consider $Y \hookrightarrow \mathcal{M}_Y$. Because of our assumptions, the functor Φ induces an open immersion $\mathcal{M}_X \hookrightarrow \mathcal{M}_Y$. Let us write U for the fibre product $X \times_{\mathcal{M}_Y} Y$.

$$\begin{tikzcd} U \arrow[rr] \arrow[d] & & Y \arrow[d] \\ X \arrow[r] & \mathcal{M}_X \arrow[r] & \mathcal{M}_Y \end{tikzcd}$$

Since the bottom row and the right column are open immersions, U is a common open subset of X and Y. By our assumptions, U contains $\kappa(x)$ and is thus nonempty. Hence, X and Y are birational. This argument is made precise in Lemma 3.4 below.

Moduli

To give a rigorous proof, we must clarify what $\mathcal{M}_X$ is. Here we mean Inaba's moduli space of simple objects [**11**] of the derived category. The advantage of using this moduli space (as opposed to the derived stack of Toën and Vaquié [**20**] or Lieblich's 'mother of all moduli' [**14**]) is that $\mathcal{M}_X$ is an *algebraic space*. Thus one need not worry about automorphisms (nor derived schemes).

Two remarks

For the reader who typically only contemplates algebraically closed fields, a potential confusion might arise.

Remark 1.4. Let S be X, but viewed as a moduli space of skyscraper sheaves. Corollary 3.2 states that $S \subset \mathcal{M}_X$ is an open immersion.

Let now $x \in X(k)$ be a closed point, with $k \neq \mathtt{k}$. The skyscraper sheaf $\kappa(x)$ is a perfectly good coherent sheaf (flat over $\mathtt{k}$), hence it defines a point $\kappa(x) \in \mathcal{M}_X(\mathtt{k})$. Since $\kappa(x)$ is a skyscraper, one might be tempted to say that $\kappa(x) \in S(\mathtt{k}) \subset \mathcal{M}_X(\mathtt{k})$. But $S = X$ and $x \notin X(\mathtt{k})$!

The issue is of course quite simple. Let X_k be the base change of X to K, with projection $\alpha\colon X_k \to X$. What we have been calling $\kappa(x)$ is actually $\alpha_*\kappa(x')$, where x' is x but viewed as a point of X_k. Essentially, we are taking k and viewing it as a $\mathtt{k}$-vector space. This clarifies our use of Φ_k in place of Φ in the statement of our theorem.

Remark 1.5. Suppose $E \in D(X)$, $F \in D(Y)$ are simple sheaves and that $\Phi(E) \simeq F$. Let $k = \mathrm{Hom}_X(E, E) = \mathrm{Hom}_Y(\Phi(E), \Phi(E)) = \mathrm{Hom}_Y(F, F)$. This implies $E \in \mathcal{G}_X(k)$ and $F \in \mathcal{G}_Y(k)$. More rigorously, let $\alpha\colon X_k \to X$ and $\beta\colon Y_k \to Y$ be the projections. There exist $E' \in D(X_k)$, $F' \in D(Y_k)$ such that $\alpha_* E' \simeq E$ and $\beta_* F' \simeq F$.

A simple base change computation shows that $\Phi(E) = \beta_* \Phi_k(E')$. Thus, an isomorphism $\Phi_k(E') \simeq F'$ (which is what we assume in our theorem) is equivalent to the data of an isomorphism $\Phi(E) \simeq F$ compatible with the extra k-linear structure. This is entirely parallel to the following situation. Let R be a $\mathtt{k}$-algebra and suppose M, N are $R \otimes_{\mathtt{k}} k$-modules. An isomorphism $M \simeq N$ of $R \otimes_{\mathtt{k}} k$-modules is the same as a k-linear isomorphism $M \simeq N$ of R-modules.

Literature

At least in the case where Φ is an equivalence (and X and Y are smooth), the first theorem is well known to experts. For example, it appears as (an easy consequence of) [**9**, Corollary 6.14].

We should also mention an already existing relationship between $\mathrm{Pic}(X)$ and $D(X)$. What follows holds for varieties smooth and projective over an algebraically closed field of characteristic zero. Rouquier has proved that if $D(X) \simeq D(Y)$ then $\mathrm{Pic}^0(X) \times \mathrm{Aut}^0(X) \simeq \mathrm{Pic}^0(Y) \times \mathrm{Aut}^0(Y)$ [**9**, Proposition 9.45]. More intrinsically, $\mathrm{Pic}^0(X) \times \mathrm{Aut}^0(X)$ is the identity component of the group of autoequivalences of $D(X)$.

Later, Popa and Schnell [**18**] showed that if $D(X) \simeq D(Y)$ then $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$ are *isogenous*. They even go as far as to conjecture that if X and Y are derived equivalent then so are $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$, which ties in beautifully with Mukai's theorem on abelian varieties [**17**].

On a side note, it would be interesting to have a way of defining the isogeny class of $\mathrm{Pic}^0(X)$ directly from the category $D(X)$, rather than observing it is preserved under derived equivalences. One would perhaps need a refinement of the intermediate Jacobians of Marcolli and Tabuada [**16**].

Finally, Abuaf [**3**] gave an alternative proof of the invariance of the isogeny class of $\mathrm{Pic}^0(X)$, linking it to his conjecture on *homological units*.

2. *Conventions*

We fix a base field $\mathtt{k}$. All rings and spaces will be assumed to be locally of finite type over it. With the exception of Hom, all functors will be implicitly derived. If X is a space over $\mathtt{k}$ and A is a $\mathtt{k}$-algebra, we write $X_A = X \times_{\mathtt{k}} \operatorname{Spec} A$. If $A \to A_0$ is a morphism of $\mathtt{k}$-algebras and $F \in D(X_A)$ we write $F \otimes A_0$ for the pullback. If $p \in X$, we write $\kappa(p)$ for the skyscraper sheaf supported at p with stalk the residue field of p.

If X is a projective scheme, $\mathcal{M}_X$ denotes Inaba's moduli space of simple objects of the derived category [**11**, Definition 0.1]. By $D(X)$ we mean the *bounded* derived category of *coherent* sheaves on X. Moreover, the space Pic_X exists and it is a disjoint union of open quasi-projective schemes [**13**, Corollary 4.18.3].

3. *Proof*

Let X be a projecitve scheme over $\mathtt{k}$. By $\mathcal{S}_X$ we denote the moduli space of simple coherent sheaves, which is an algebraic space locally of finite type over $\mathtt{k}$. This fact can be shown as follows. There is an Artin stack, locally of finite type over $\mathtt{k}$, parameterizing coherent sheaves on X. Since being simple is an open condition in flat families of sheaves [**10**, Proposition 2.3.1], there is an open substack parameterizing simple coherent sheaves on X. The space $\mathcal{S}_X$ is obtained as the $\mathbf{G}_{\mathrm{m}}$-rigidification of the latter, in the sense of [**2**]. The following proposition appears without proof as [**11**, Remark 0.3].

Proposition 3.1. *Let X be a projective scheme over $\mathtt{k}$. The obvious morphism $\mathcal{S}_X \hookrightarrow \mathcal{M}_X$ is an open immersion.*

Proof. The map $\mathcal{S}_X \to \mathcal{M}_X$ takes a sheaf and views it as a complex sitting in degree zero. Sheaves are isomorphic if and only if they are quasi-isomorphic as complexes, hence $\mathcal{S}_X \hookrightarrow \mathcal{M}_X$ is a monomorphism. Since it is also locally of finite type, it suffices to show that it is formally smooth. In other words, given a nilpotent thickening of local Artinian $\mathtt{k}$-algebras $A \twoheadrightarrow A_0$, we must solve the following lifting problem.

$$\begin{tikzcd} \operatorname{Spec} A_0 \arrow[r] \arrow[d] & \mathcal{S}_X \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed, "\exists ?"] & \mathcal{M}_X \end{tikzcd}$$

Let k be the residue field of A. Before continuing, let us also draw the base change diagram.

$$\begin{tikzcd} X_k \arrow[r] \arrow[d] & X_{A_0} \arrow[r] \arrow[d] & X_A \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A_0 \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Let $F \in \mathcal{M}_X(A)$ be such that $F \otimes A_0 \in \mathcal{S}_X(A_0)$. We will show that F itself is in $\mathcal{S}_X(A)$.

Since $F \otimes k$ is concentrated in degree zero, it follows that F is a sheaf, flat over A (this is a standard fact about derived restriction, see [**9**, Lemma 3.31] or [**6**, Lemma 4.1]). By definition, to check $F \in \mathcal{S}_X(A)$ we must show that $F \otimes k$ is simple. But this is apparent, as $F \otimes k = (F \otimes A_0) \otimes k$ and $F \otimes A_0 \in \mathcal{S}_X(A_0)$ by construction. □

COROLLARY 3.2. *Let X be a projective scheme over $\mathtt{k}$. There are natural open immersions $X \hookrightarrow \mathcal{M}_X$ and $\operatorname{Pic}_X \hookrightarrow \mathcal{M}_X$.*

Proof. Using the proposition above, it suffices to show there are open immersions $X \hookrightarrow \mathcal{S}_X$, $\operatorname{Pic}_X \hookrightarrow \mathcal{S}_X$. The scheme X can be viewed as a moduli space of skyscraper sheaves, which is the same as the moduli space of zero dimensional stable sheaves with Hilbert polynomial 1, which is open inside $\mathcal{S}_X$ [**10**, Proposition 2.3.1]. Let us turn to Pic_X. The local criterion for flatness implies that being locally free is an open condition in flat families of sheaves, hence the claim. □

The first part of the corollary, at least when X is smooth, can be deduced from [**15**, Lemma 5.2] (we thank Daniel Huybrechts for pointing this out).

3.1. *Functors*

Let now X, Y be Gorenstein projective schemes over $\mathtt{k}$. Let $\Phi\colon D(X) \to D(Y)$ be an integral transform defined by a kernel K of finite Tor-dimension over both X and Y. The Gorenstein assumption ensures Φ has a left adjoint Ψ, also defined by a finite Tor-dimensional kernel [**8**, Proposition 1.17].

If A is a $\mathtt{k}$-algebra, the base change K_A defines a functor $\Phi_A\colon D(X_A) \to D(Y_A)$. Moreover, A-perfect complexes of $D(X_A)$ are sent to A-perfect complexes of $D(Y_A)$. Therefore we have a well-defined morphism $\mathcal{M}_X \to \mathcal{M}_Y$ (implicitly we are using the results of [**19**, Exposé III]).

PROPOSITION 3.3. *In the setting above, assume moreover Φ to be fully faithful. Then $\mathcal{M}_X \to \mathcal{M}_Y$ is an open immersion.*

Proof. Once more, we will prove that $\mathcal{M}_X \to \mathcal{M}_Y$ is a formally smooth monomorphism. Let Ψ be the left adjoint of Φ (one might as well work with the right adjoint). As Φ is fully faithful, the counit $\mathrm{id} \to \Psi \circ \Phi$ is an isomorphism. In terms of kernels, we have an isomorphism $\mathcal{O}_{\Delta_X} \simeq L * K$, where L is representing Ψ. Finally, if $\operatorname{Spec} A \to \mathtt{k}$ is any map, we also have an isomorphism $\mathcal{O}_{\Delta_{X_A}} \simeq (L \otimes A) * (K \otimes A)$. In particular, this shows that $\mathcal{M}_X \to \mathcal{M}_Y$ is a monomorphism.

To prove formal smoothness, let $A \twoheadrightarrow A_0$ be a nil-thickening of local Artinian $\mathtt{k}$-algebras. Let $E \in \mathcal{M}_Y(A)$ be such that $E \otimes A_0 \simeq \Phi_A(F)$ for $F \in \mathcal{M}_X(A_0)$. Consider $\Psi_A(E)$. We have an exact triangle

$$E \to \Phi_A \Psi_A(E) \to C \to E[1],$$

where the first map is the unit of the adjunction. Base changing to A_0, we have

$$\Phi(F) \to \Phi_{A_0}\Psi_{A_0}\Phi_{A_0}(F) \to C \otimes A_0 \to \Phi(F)[1]$$

and, since $\Psi_{A_0}\Phi_{A_0}$ is isomorphic to the identity, the first map is an isomorphism. Hence $C \otimes A_0 = 0$. It follows that the support of C is empty, hence $C = 0$. This means $E \simeq \Phi_A\Psi_A(E)$ and thus $\mathcal{M}_X \to \mathcal{M}_Y$ is an open immersion. □

Proof of the main theorem

Now that all the technicalities are out of the way, the main result follows easily.

LEMMA 3.4. *Let X and Y be two geometrically integral, Gorenstein and projective varieties over a field $\mathtt{k}$. Let $\Phi\colon D(X) \to D(Y)$ be a fully faithful integral transform defined by a kernel of finite Tor-dimension over both X and Y. Let $M \hookrightarrow \mathcal{M}_X$ and $N \hookrightarrow \mathcal{M}_Y$ be two open immersions, with M and N irreducible schemes. Finally, let $m \in M(k)$ correspond to a complex $E \in D(X_k)$ and let $n \in N$ correspond to a complex $F \in D(Y_k)$. Assume there exists an isomorphism $\Phi_k(E) \simeq F$. Then M and N are birational.*

Proof. As explained in the introduction, consider the fibre product $U = M \times_{\mathcal{M}_Y} N$. This is a common open subscheme of M and N. Since $m \in U(k)$, it is nonempty. □

Theorem 1.1 and Corollaries 1.2, 1.3 now follow.

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References

1. D. Abramovich and J.-C. Chen, 'Flops, flips and perverse point sheaves on threefold stacks', *J. Algebra* 290 (2005) 372–407.
2. D. Abramovich, A. Corti and A. Vistoli, 'Twisted bundles and admissible covers', *Comm. Algebra* 31 (2003) 3547–3618, Special issue in honor of Steven L. Kleiman.
3. R. Abuaf, 'Homological units', Preprint, 2015, arXiv:1510.01583.
4. A. Bondal and D. Orlov, 'Semiorthogonal decomposition for algebraic varieties', Preprint, 1995, arXiv:alg-geom/9506012.
5. T. Bridgeland, 'Flops and derived categories', *Invent. Math.* 147 (2002) 613–632.
6. J. Calabrese, 'Relative singular twisted Bondal–Orlov', *Math. Res. Lett.*, to appear.
7. J.-C. Chen, 'Flops and equivalences of derived categories for threefolds with only terminal Gorenstein singularities', *J. Differential Geom.* 61 (2002) 227–261.
8. D. Hernández Ruipérez, A. C López Martín and F. S. de Salas, 'Fourier-Mukai transforms for Gorenstein schemes', *Adv. Math.* 211 (2007) 594–620.
9. D. Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Mathematical Monographs (Clarendon Press, Oxford University Press, Oxford, 2006) viii+307 pp.
10. D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, 2nd edn (Cambridge Mathematical Library, Cambridge, 2010).
11. M. Inaba, 'Toward a definition of moduli of complexes of coherent sheaves on a projective scheme', *Koyoto J. Math* 42 (2002) 317–329.

12. Y. Kawamata, 'Flops connect minimal models', *Publ. Res. Inst. Math. Sci.* 44 (2008) 419–423.
13. S. Kleiman, 'The Picard scheme', Preprint, 2005, arXiv:math/0504020.
14. M. Lieblich, 'Moduli of complexes on a proper morphism', *J. Algebraic Geom.* 15 (2006) 175–206.
15. M. Lieblich and M. Olsson, 'Fourier-Mukai partners of K3 surfaces in positive characteristic', *Ann. Sci. Éc. Norm. Supér.* (4) 48 (2015) 1001–1033.
16. M. Marcolli and G. Tabuada, 'Jacobians of noncommutative motives', *Mosc. Math. J.* 14 (2014) 577–594, 642.
17. S. Mukai, 'Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves', *Nagoya Math. J.* 81 (1981) 153–175.
18. M. Popa and C. Schnell, 'Derived invariance of the number of holomorphic 1-forms and vector fields', *Ann. Sci. Éc. Norm. Supér.* (4) 44 (2011) 527–536.
19. *Thorie des intersections et thorme de Riemann-Roch*, Sminaire de Gomtrie Algbrique du Bois-Marie 1966-1967 (SGA 6). Dirig par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, Michel Raynaud et J. P. Serre, Lecture Notes in Mathematics 225 (Springer, Berlin/New York, 1971).
20. B. Toën and M. Vaquié, 'Moduli of objects in dg-categories', *Ann. Sci. Éc. Norm. Supér.* (4) 40 (2007) 387–444.
21. M. Van den Bergh, 'Three-dimensional flops and noncommutative rings', *Duke Math. J.* 122 (2004) 423–455.

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