## Fort Collins Feb 2014

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OK, this is why I got a PhD.
Theorem (my advisor was Bridgeland) - If Y, W are two birational smooth and projective Calabi-Yau threefolds (over $\mathbb{C}$ ), then there is a precise relationship between their Donaldson-Thomas invariants. Explicitly, if $\mathrm{Y} \rightarrow \mathrm{X}$ and $\mathrm{W} \rightarrow \mathrm{X}$ are related by a flop then we have a pretty formula like this.

$$
\mathrm{DT}(\mathrm{Y} / \mathrm{X})=\mathrm{DT}(\mathrm{~W} / \mathrm{X})
$$

There is also a similar looking formula if we allow orbifolds, but I don't think I'll be able to get to that. (this would be the setting of the McKay correspondence and goes by the name of the crepant resolution conjecture)

By the end of the seminar the statement will hopefully become clearer. The proof goes via derived categories, so I guess I should say something about those as well.

## Donaldson-Thomas Invariants

(good reference for this: Pandharipande-Thomas $13 / 2$ ways to count curves) We want X to be a CalabiYau threefold:

- smooth, connected, projective variety over $\mathbb{C}$,
$-\omega_{\mathrm{X}} \cong \mathcal{O}_{\mathrm{X}}$,
$-\mathrm{H}^{1}\left(\mathrm{X}, \mathcal{O}_{\mathrm{X}}\right)=0$.
(the last one is equivalent to saying X has torsion fundamental group)
Enumerative geometry is popular these days. We might say that the point is to try and "count" something about X . A lot of focus is on curves. The naive idea would be
- take some moduli space $\mathrm{M}_{\mathrm{X}}$ of geometric objects attached to X ,
- perhaps $\mathrm{M}_{\mathrm{X}}$ was defined from a larger moduli space by imposing additional constraints (like tangency or stability conditions),
- if all went well, $\mathrm{M}_{\mathrm{X}}$ is a compact zero-dimensional variety and we count how many points it has,

$$
\operatorname{Invariant}(\mathrm{X})=\# \mathrm{M}_{\mathrm{X}}=\chi_{t o p}\left(\mathrm{M}_{\mathrm{X}}\right),
$$

- in practice - this never happens.

What actually happens is

- the study of the deformation theory of $\mathrm{M}_{\mathrm{X}}$ yields a special homology class $\left[\mathrm{M}_{\mathrm{X}}\right]_{\text {vir }}$ called a virtual fundamental class (it's not intrinsic to $\mathrm{M}_{\mathrm{X}}$ - work of Schürg showed a relationship between virtual classes and the possible ways in which $\mathrm{M}_{\mathrm{X}}=\pi_{0}(\widetilde{\mathrm{M}})$, for a derived scheme $\left.\widetilde{\mathrm{M}}\right)$,
- not always, but sometimes, $\left[\mathrm{M}_{\mathrm{X}}\right]_{\text {vir }}$ is a zero-cycle,
- integrating $\int_{\left[\mathrm{M}_{\mathrm{X}}\right]_{\text {vir }}} 1$ (or taking its degree) we obtain a number,
- we call this number the virtual count of whatever we were counting and call it a day.

Donaldson-Thomas theory begins with Richard Thomas's thesis and has seen several extensions and variants (Joyce-Song, Kontsevich-Soibelman and others). The flavour we are interested in has to do with counting curves.

Before DT theory came Gromov-Witten theory. In GW one "counts" maps

$$
\mathrm{C} \rightarrow \mathrm{X}
$$

where C is a nice (at worst nodal) curve. The maps however can contract components and one needs to worry about automorphisms. So, all in all it's not super nice. The relevant moduli space is Kontsevich's stack of stable maps.

In DT theory the perspective is switched: the maps are as nice as they get (inclusions), but the curves are impossibly bad. The relevant moduli space is $\mathrm{M}_{\mathrm{X}}=\mathrm{Hilb}_{\mathrm{X}}$ - the Hilbert scheme of X

$$
\operatorname{Hilb}_{\mathrm{X}}=\{\mathrm{Z} \hookrightarrow \mathrm{X}\}=\left\{\mathcal{O}_{\mathrm{X}} \rightarrow \mathcal{O}_{\mathrm{Z}}\right\}
$$

which parameterises subschemes of X or, equivalently, quotients of the structure sheaf.
There is actually a deep (and complicated!) relationship between DT and GW, going by the name of the MNOP conjectures (which have been proven for many things, I think including everything toric). There are other counting theories. As it's related to this we should mention Pandharipande-Thomas theory (which is similar to DT theory but where one does not allow points to float off the curve). Anyway, let's go back to DT theory.

As usual, if we fix some topological data (namely the Hilbert polynomial or the Chern character of $\mathcal{O}_{\mathrm{Z}}$ ) the moduli space splits into open and closed components

$$
\operatorname{Hilb}_{\mathrm{X}}=\coprod_{\alpha} \operatorname{Hilb}_{\mathrm{X}}(\alpha)=\left\{\mathcal{O}_{\mathrm{X}} \rightarrow \mathcal{O}_{\mathrm{Z}} \mid \operatorname{ch}\left(\mathcal{O}_{\mathrm{Z}}\right)=\alpha\right\} .
$$

We are interested in $\alpha=(0,0, \beta, n)$, so that $\beta$ is the homology class corresponding to $[\mathrm{Z}]$ and by Riemann-Roch $n$ is the holomorphic Euler characteristic of Z

$$
n=\sum_{i}(-1)^{i} \operatorname{dim}_{\mathbb{C}} \mathrm{H}^{i}\left(\mathrm{Z}, \mathcal{O}_{\mathrm{Z}}\right) .
$$

Notice that we also allow $\beta=0$.
However, the correct approach to describe the virtual class is by considering HilbX as a moduli space of ideal sheaves (which are torsion free sheaves with trivial determinant). However again, for DT theory for CY3's a miracle happens.
"Instead of using complicated homology classes, one uses complicated Euler characteristics."

Kai Behrend has shown that every scheme of finite type M comes with a constructible function $\nu_{\mathrm{M}}: \mathrm{M}(\mathbb{C}) \rightarrow \mathbb{Z}$ (called its Behrend function). When M is smooth of dimension $d$ then $\nu_{\mathrm{M}} \equiv(-1)^{d}$ is constant. This function is deformation invariant and thus picks up on the possible non-reduced structure of M (e.g. two points coming together to form a fat point).

Given the Behrend function we can define the weighted Euler characteristic of M as

$$
\chi_{\mathrm{B}}(\mathrm{M}):=\sum_{k} k \cdot \chi_{t o p}\left[\nu_{\mathrm{M}}^{-1}(k)[.\right.
$$

We can think of the Euler characteristic as a substitute for counting when a space is not zerodimensional. This weighted Euler characteristic is an even fancier version (actually, the theorem also holds for ordinary Euler characteristic as well).

Theorem (Behrend) - For X a CY3,

$$
\int_{\left[\operatorname{Hilb}_{\mathrm{X}}(\beta, n)\right]_{v i r}} 1=\chi_{\mathrm{B}}\left(\operatorname{Hilb}_{\mathrm{X}}(\beta, n)\right)=\mathrm{DT}_{\mathrm{X}}(\beta, n)
$$

It is customary to bundle everything up in a generating series.

$$
\mathrm{DT}(\mathrm{X})=\sum_{\beta, n} \mathrm{DT}_{\mathrm{X}}(\beta, n) q^{(\beta, n)}
$$

We think of $\mathrm{DT}(\mathrm{X}) \in \mathrm{R}$ as living in some ring R of formal series. The definition of R contains a subtlety, in the sense that we need to consider not just finite sums, but infinite ones (this is related to the fact that Hilb is only locally of finite type but it is indeed not quasi-compact). We'll completely ignore this.

Anyway, the upshot of all this is that if you don't like virtual classes or don't like this flavour of enumerative geometry altogether, you can just take

$$
\overline{\mathrm{DT}}_{\mathrm{X}}(\beta, n)=\chi_{t o p}\left(\operatorname{Hilb}_{\mathrm{X}}(\beta, n)\right)
$$

which we might call the "unweighted" or naive DT numbers (which unfortunately won't be deformation invariant anymore). In what follows I could take either one.
Flops

Theorem - Two birational Calabi-Yaus are connected by a chain of flops. (the general version of this is due to Kawamata - I think in the setting we need it was already shown by Kollar)

What is a flop? OK, the standard flop is the following. Assume Y contains a $(-1,-1)$-curve C - this just means that $\mathrm{C}=\mathbb{P}^{1}$ and that the normal bundle of C in Y is $\mathcal{O}_{\mathrm{C}}(-1) \oplus \mathcal{O}_{\mathrm{C}}(-1)$. Let's blow up the curve: $\mathrm{Z}=\mathrm{B} l_{\mathrm{C}} \mathrm{Y}, \pi: \mathrm{Z} \rightarrow \mathrm{Y}$. The exceptional locus is the projectivisation of that bundle and thus is isomorphic to $\mathbb{P}^{1} \times \mathbb{P}^{1}$. Instead of contracting via $\pi$, we collapse in the other direction, obtaining $\rho: \mathrm{Z} \rightarrow \mathrm{W}$. This W is called the flop of Y along C .


Actually one does a different thing. We start with $\mathrm{Y} \rightarrow \mathrm{X}$ a surjective birational morphism, where Y is our Calabi-Yau threefold (in the example above we would be contracting C instead of blowing it up). The locus $\mathrm{E} x$ which is contracted by $\mathrm{Y} \rightarrow \mathrm{X}$ is assumed to be one-dimensional. The flop of Y (along $\mathrm{E} x$ ) is any other $\mathrm{W} \rightarrow \mathrm{X}$, satisfying the same assumptions as $\mathrm{Y} \rightarrow \mathrm{X}$, and such that the proper transform of any relatively anti-ample divisor is relatively ample (for this stuff you can have a look here).

Two nice things about this.
Theorem (Bridgeland) - The derived categories of $Y$ and $W$ are the same $-\mathrm{D}(\mathrm{Y}) \simeq \mathrm{D}(\mathrm{W})$. Actually, there is a heart $\operatorname{Per}(\mathrm{Y} / \mathrm{X}) \subset \mathrm{D}(\mathrm{Y})$ of a t -structure in $\mathrm{D}(\mathrm{Y})$ and W is a moduli space of point-objects in $\mathrm{P} e r(\mathrm{Y} / \mathrm{X})$. Moreover, $\mathrm{P} e r(\mathrm{Y} / \mathrm{X})=\mathrm{P} e r(\mathrm{~W} / \mathrm{X}) \boldsymbol{I}^{1}$

## Coherent Sheaves

OK, for some reason I want to start by this completely unrelated result.
Theorem (Gabriel and later Rosenberg, Gabber/Brandenburg, Myself-Groechenig) - ${ }^{2}$ A variety is completely recovered by its category of coherent sheaves.

What do we mean by this? If $\mathrm{X}, \mathrm{Y}$ are two varieties such that $\operatorname{Coh}(\mathrm{X}) \simeq \operatorname{Coh}(\mathrm{Y})$ then $\mathrm{X} \simeq \mathrm{Y}$. Why do we care? Geometry is hard, se we would like to linearise it and do some (hopefully easier!) algebra. $\operatorname{Coh}(\mathrm{X})$ is a linear category but it's too strong as it remembers everything about X . We can think of $\mathrm{D}(\mathrm{X})$ as a compromise: it's still linear enough (we can do homological algebra) but more flexibility is allowed.

Derived categories were born in an attempt by Verdier to generalise Poincaré and Serre duality. But even without needing such advanced technology we might say that, from a modern point of view, homological algebra should start with derived categories as $\mathrm{D}(\mathrm{X})$ is the home of all derived functors. Let's recall how.

## Derived Categories

What do we like about $\operatorname{Coh}(\mathrm{X})$ ? It's an abelian category, which means:

- it's additive - so we can sum morphisms together and there is a zero object;
- it's abelian - we can take kernels, cokernels, images and pretend we are working in a category of modules;
- in particular we can talk about short (and long) exact sequences.

If $\mathcal{A}$ is an abelian category, it makes sense to talk about chain complexes:

$$
\cdots \rightarrow \mathrm{E}^{n-1} \rightarrow \mathrm{E}^{n} \rightarrow \mathrm{E}^{n+1} \rightarrow \cdots
$$

[^0]which are strings of morphisms such that the composition of two of them is zero. Such gadgets come with cohomology objects
$$
\mathrm{H}^{n}(\mathrm{E})=\frac{\operatorname{ker}^{n} \mathrm{E}^{n} \rightarrow \mathrm{E}^{n+1}}{\operatorname{imE}^{n-1} \rightarrow \mathrm{E}^{n}}
$$
for $n \in \mathbb{Z}$. A quasi-isomorphism is a map of chain complexes $\varphi: \mathrm{E} \rightarrow \mathrm{F}$ inducing an isomorphism on cohomology $H^{n}(E) \simeq H^{n}(F)$.
Definition 1.1 - Let $\mathcal{A}$ be an abelian category. We define $\mathrm{D}(\mathcal{A})=\operatorname{Ch}(\mathcal{A})\left[q i s^{-1}\right]$.
So the objects are chain complexes but the morphisms are obtained by formally inverting all quasiisomorphisms. Notice that $\mathcal{A}$ sits inside $\mathrm{D}(\mathcal{A})$ by sending A to the complex with zeros everywhere except in degree zero.

But why the hell would one want to do this? Well, how do you compute sheaf cohomology $\mathrm{H}^{\bullet}(\mathrm{X},-)$ ? It's the derived functor of global sections $\Gamma(\mathrm{X},-)$. We take a sheaf F - pick an appropriate (say, injective) resolution

$$
\mathrm{F} \rightarrow \mathrm{I}_{0} \rightarrow \mathrm{I}_{1} \rightarrow \cdots
$$

By applying $\Gamma$ we obtain a chain complex

$$
\mathrm{R} \Gamma(\mathrm{X}, \mathrm{~F})=\cdots \rightarrow 0 \rightarrow \Gamma\left(\mathrm{X}, \mathrm{I}_{0}\right) \rightarrow \Gamma\left(\mathrm{X}, \mathrm{I}_{1}\right) \rightarrow \cdots
$$

by taking cohomology of the complex

$$
\mathrm{H}^{i}(\mathrm{X}, \mathrm{~F})=\mathrm{H}^{i}(\mathrm{R} \Gamma(\mathrm{X}, \mathrm{~F}))=\frac{\operatorname{ker}\left(\Gamma\left(\mathrm{X}, \mathrm{I}_{i}\right) \rightarrow \Gamma\left(\mathrm{X}, \mathrm{I}_{i+1}\right)\right)}{\mathrm{im}\left(\Gamma\left(\mathrm{X}, \mathrm{I}_{i-1}\right) \rightarrow \Gamma\left(\mathrm{X}, \mathrm{I}_{i}\right)\right)}
$$

we get our sheaf cohomology. We like the whole complex $R \Gamma(X, F)$ is it packaged all the information about sheaf cohomology of F. For example, whenever we want to say about all cohomology sheaves at once we can just say something about $R \Gamma(\mathrm{X}, \mathrm{F})$. However: what happens if we change the resolution? The complex changes, although the cohomology doesn't. So, in this sense, the notation $\mathrm{R} \Gamma(\mathrm{X}, \mathrm{F})$ is ill-defined. But, if you notice F is quasi-isomorphic to I . In $\mathrm{D}(\mathrm{X})$ all the relevant complexes become isomorphic and $\mathrm{R} \Gamma(\mathrm{X}, \mathrm{F})$ is indeed well-defined.

The drawback here is that $\mathrm{D}(\mathrm{X})$ is no longer abelian, it will be triangulated. This means

- it's additive;
- it has a shift functor [1] (take a complex and shift it all to the left);
— we have mapping cones (which play the role of ker and coker combined);
- instead of short exact sequences we have distinguished triangles.

If you like topology, you might think of chain complexes as being complexes $\mathrm{C}_{*}(\mathrm{X})$ of singular chains on a space (or spectrum) X. Suspending a space $\Sigma \mathrm{X}$ corresponds to shifting the complex of chains. If $f: \mathrm{X} \rightarrow \mathrm{Y}$ is a map of topological spaces, we define the mapping cone $\mathrm{C}_{f}$ by taking X , crossing it with the unit interval I , collapsing $\mathrm{X} \times\{0\}$ to a point and gluing $\mathrm{X} \times\{1\}$ onto Y via $f$. The complex $\mathrm{C}_{*}\left(\mathrm{C}_{f}\right)$ is homotopy-equivalent to the mapping cone of $\mathrm{C}_{*}(\mathrm{X}) \rightarrow \mathrm{C}_{*}(\mathrm{Y})$.

Anyhow, we say that a sequence $\mathrm{A} \rightarrow \mathrm{B} \rightarrow \mathrm{C} \rightarrow \mathrm{A}[1]$ in the derived category is a distinguised triangle if (more or less) it's isomorphic to a triangle $\mathrm{A}^{\prime} \rightarrow \mathrm{B}^{\prime} \rightarrow \mathrm{C}_{f}\left(\mathrm{~A}^{\prime} \rightarrow \mathrm{B}^{\prime}\right) \rightarrow \mathrm{A}^{\prime}[1]$. We call $\mathrm{A}, \mathrm{B}, \mathrm{C}$ the vertices of the triangle.

All the functors we like extend to the derived category:

$$
\mathrm{H}^{i}(\mathrm{X},-) \leadsto \mathrm{R} \Gamma(\mathrm{X},-) ; \mathrm{R}^{i} f_{\star} \leadsto \mathrm{R} f_{\star} ; \operatorname{Tor}_{i} \leadsto \otimes^{\mathrm{L}} ; \mathrm{Ext}^{i} \leadsto \mathrm{RHom} ;
$$

and similarly derived pullback $\mathrm{L} f^{*}$ and local hom RHom . Actually we also get a new one (totalling five of the infamous six operations), $f$ ! the 'shriek' pullback - this is the functor Verdier was looking for to generalise Poincaré and Serre duality.

When $f$ is proper, this is the right adjoint of $\mathrm{R} f_{\star}$. Given $\mathrm{F} \in \operatorname{Coh}(\mathrm{X})$ we can think of it as a complex sitting in degree zero, and actually $\operatorname{Coh}(\mathrm{X}) \subset \mathrm{D}(\mathrm{X})$ is fully faithful and sends short exact sequences to distinguished triangles. On the other hand, $\mathrm{H}^{i}(\mathrm{E})$ sends a complex to its $i$-th cohomology, so defined a functor $\mathrm{D}(\mathrm{X}) \rightarrow \mathrm{Coh}(\mathrm{X})$. This has the nice feature that if $\mathrm{A} \rightarrow \mathrm{B} \rightarrow \mathrm{C} \rightarrow \mathrm{A}[1]$ is a distinguished triangle, then

$$
\cdots \mathrm{H}^{i-1}(\mathrm{C}) \rightarrow \mathrm{H}^{i}(\mathrm{~A}) \rightarrow \mathrm{H}^{i}(\mathrm{~B}) \rightarrow \mathrm{H}^{i}(\mathrm{C}) \rightarrow \mathrm{H}^{i+1}(\mathrm{~A}) \cdots
$$

is a long exact sequence in $\operatorname{Coh}(\mathrm{X})$. Finally, the Ext functors become particularly nice as well: $\mathrm{Ext}{ }^{i}(\mathrm{~A}, \mathrm{~B})=\mathrm{Hom}_{\mathrm{D}(\mathrm{X})}(\mathrm{A}, \mathrm{B}[i])$.

Understanding what $\mathrm{D}(\mathrm{X})$ remembers about X is subtle, but here's a couple.
Theorem 1.2 - Let $X, Y$ be two smooth and projective varieties with $D(X) \simeq D(Y)$ then:
$-\operatorname{dim}(\mathrm{X})=\operatorname{dim}(\mathrm{Y})$
$-\chi_{t o p}(\mathrm{X})=\chi_{t o p}(\mathrm{Y})$

- the odd and even parts of rational cohomology are preserved (but the grading gets jumbled up!)
$-\operatorname{kod}(\mathrm{X})=\operatorname{kod}(\mathrm{Y})$
- if $\operatorname{dim}(\mathrm{X})=1$ then $\mathrm{Y} \simeq \mathrm{X}$
— if $\operatorname{dim}(\mathrm{X})=3$ then $h^{p, q}(\mathrm{X})=h^{p, q}(\mathrm{Y})$
- $\pi_{1}$ is not derived invariant, even for threefolds
- if $\omega_{\mathrm{X}}^{ \pm 1}$ is ample then $\mathrm{X} \simeq \mathrm{Y}$.

OK, let's go back to flops for a second. The category $\operatorname{Per}(\mathrm{Y} / \mathrm{X})$ is more or less defined as follows.

$$
\operatorname{Per}(\mathrm{Y} / \mathrm{X})=\left\{\mathrm{E} \in \mathrm{D}(\mathrm{Y}) \mid \mathrm{E}^{n}=0, n \neq-1,0 ;\right.
$$

$$
\left.f_{\star} \mathrm{H}^{-1}(\mathrm{E})=0=\mathrm{R}^{1} f_{\star} \mathrm{H}^{0}(\mathrm{E})\right\}
$$

So, if $\mathrm{E} \in \operatorname{Coh}(\mathrm{X}), \operatorname{supp}(\mathrm{E}) \cap \mathrm{E} x=\varnothing$, then $\mathrm{E} \in \mathrm{P} e r$. If $\mathrm{C}=\mathrm{P}^{1}$ is contracted by $f$, then $\mathcal{O}_{\mathrm{C}}(-2) \notin \mathrm{Per}$, but $\mathcal{O}_{\mathrm{C}}(-2)[1] \in \mathrm{Per} \square^{3}$

This subcategory is special inside $\mathrm{D}(\mathrm{X})$, it's a so-called heart of a t -structure and possesses a few features in common with Coh(X). It is an abelian category - where a short exact sequence is a distinguished triangle whose vertices all lie in Per. Being an inclusion, $\mathrm{Per} \rightarrow \mathrm{D}(\mathrm{X})$ is fully faithful and exact. There are also "cohomology functors" $\mathrm{H}_{\mathrm{P} e r}^{i}: \mathrm{D}(\mathrm{Y}) \rightarrow \mathrm{P} e r$, but we won't need them.

OK, so Per gives us a way to draw a bridge (no pun intended) between Y and W , but it's still unclear how to employ it for DT invariants. This is where Hall algebras come in.

[^1]
## Hall Algebras

The rough idea is as follows. Take $\mathcal{A}$ to be an abelian category. We want to define an algebra $\mathrm{H}(\mathcal{A})$, where the elements are functions

$$
f: \mathcal{A} \rightarrow \mathbb{Q}
$$

so to each object (or, rather, isomorphism class) $\mathrm{A} \in \mathcal{A}$ corresponds a rational number $f(\mathrm{~A})$. The algebra structure is given by a "convolution" product $f * g$ where

$$
f * g(\mathrm{~B})=\int_{\mathrm{E} x t_{\mathrm{B}}^{1}} f\left(\mathrm{~A}_{2}\right) g\left(\mathrm{~A}_{1}\right)=\sum_{0 \rightarrow \mathrm{~A}_{1} \rightarrow \mathrm{~B} \rightarrow \mathrm{~A}_{2} \rightarrow 0} f\left(\mathrm{~A}_{1}\right) g\left(\mathrm{~A}_{1}\right)=\sum_{\mathrm{A} \rightarrow \mathrm{~B}} f(\mathrm{~A}) g(\mathrm{~B} / \mathrm{A}) .
$$

When for example $\mathcal{A}=\operatorname{Coh}\left(\mathbb{P}_{\mathbb{F}_{q}}^{1}\right)$ and we take only functions with finite support then $\mathrm{H}(\mathcal{A})$ makes perfect sense. Sadly, for $\operatorname{Coh}(\mathrm{Y})$ our CY3 this does not make sense, we need something fancier. Nevertheless, there are some valuable ideas knocking about. For example, if $\mathcal{B} \subset \mathcal{A}$ is a subcategory we can define $1_{\mathcal{B}}$ as

$$
1_{\mathcal{B}}(E)= \begin{cases}1 & E \in \mathcal{B} \\ 0 & E \notin \mathcal{B}\end{cases}
$$

Notice that, for $\mathcal{B}=\{0\}, 1_{0}$ is the unit of $*$.
If, say, we have two subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$, with the property that any $\mathrm{E} \in \mathcal{A}$ sits in a unique short exact sequence

$$
0 \rightarrow \mathrm{~T}_{\mathrm{E}} \rightarrow \mathrm{E} \rightarrow \mathrm{~F}_{\mathrm{E}} \rightarrow 0
$$

with $\mathrm{T}_{\mathrm{E}} \in \mathcal{T}, \mathrm{F}_{\mathrm{E}} \in \mathcal{F}$ then

$$
1_{\mathcal{A}}=1_{\mathcal{T}} * 1_{\mathcal{F}}
$$

(such a thing is called a torsion pair).
If $\mathcal{O} \in \mathcal{A}$ is some element, we have $1_{\mathcal{B}}^{\mathcal{O}} \in \mathrm{H}(\mathcal{A})$ defined as

$$
1_{\mathcal{B}}^{\mathcal{O}}(\mathrm{E})= \begin{cases}0 & \mathrm{E} \notin \mathcal{B} \\ \#\{\mathcal{O} \rightarrow \mathrm{E}\}=\# \operatorname{Hom}(\mathcal{O}, \mathrm{E}) & \mathrm{E} \in \mathcal{B}\end{cases}
$$

Moreover, we have a sort of "quot"-element, $Q^{\mathcal{O}}$ defined as

$$
\mathrm{Q}^{\mathcal{O}}(\mathrm{E})=\#\{\mathcal{O} \rightarrow \mathrm{E}\}
$$

As $\mathcal{A}$ is an abelian category, it has a first isomorphism theorem, which translates into the identity

$$
1_{\mathcal{A}}^{\mathcal{O}}=\mathrm{Q}^{\mathcal{O}} * 1_{\mathcal{A}} .
$$

Why? Well, it makes sense. Given $\mathcal{O} \rightarrow \mathrm{E}$ we can take its image $\mathcal{O} \rightarrow \mathrm{I}$, which fits into a short exact sequence $\mathrm{I} \rightarrow \mathrm{E} \rightarrow \mathrm{E} / \mathrm{I}$. On the other hand, given such an extension+surjection we can just compose $\mathcal{O} \rightarrow \mathrm{I} \rightarrow \mathrm{E}$.

So, how to make this work for $\operatorname{Coh}(Y)$ ? In some sense we just substitute \# with $\chi$.
First we need the Grothendieck ring of varietis $\mathrm{K}_{0}(\mathrm{~V}$ ar $)$ : it is the free abelian group spanned be $[\mathrm{X}]$, for X a variety, modded out by the scissor (or cut $\mathcal{E}$ paste) relations. These relations say that, if $\mathrm{Y} \rightarrow \mathrm{X}$ is closed then

$$
[\mathrm{X}]=[\mathrm{Y}]+[\mathrm{X}, \mathrm{Y}] .
$$

Disjoint union becomes a sum $[\mathrm{X} \sqcup \mathrm{Y}]=[\mathrm{X}]+[\mathrm{Y}]$ and product endows $\mathrm{K}_{0}(\mathrm{~V} a r)$ with a ring structure $[\mathrm{X} \times \mathrm{Y}]=[\mathrm{X}] \cdot[\mathrm{Y}]$.

We should think of $\mathrm{K}_{0}(\mathrm{~V} a r)$ as the universal recipient of "additive" invariants: any invariant of varieties which is additive under stratifications must factor through this ring. In particular, the topological Euler characteristic factors through it (but also the Poincaré or Hodge polynomials).

We need something even more general: the Grothendieck of stacks, $\mathrm{K}_{0}(\mathrm{~S} t)$ which is spanned by $[\mathrm{X}]$ for X a finite-type Artin stack with affine diagonal (again, just as for our series ring R, finite-typeness will get me into trouble and I'll cheat). Actually, we need a relative version of all this. Fix a base S, we consider $[X \rightarrow S]$ with the relations $Y \rightarrow X$ then

$$
[\mathrm{X} \rightarrow \mathrm{~S}]=[\mathrm{Y} \rightarrow \mathrm{X} \rightarrow \mathrm{~S}]+[\mathrm{X} \backslash \mathrm{Y} \rightarrow \mathrm{X} \rightarrow \mathrm{~S}]
$$

Now let's take the base $S$ to be the stack of coherent sheaves

$$
\operatorname{Coh}(\mathrm{Y})=\underline{\operatorname{Coh}}(\mathrm{Y})=\mathcal{M}=\{\mathrm{E} \in \operatorname{Coh}(\mathrm{Y})\} .
$$

As usual, by fixing chern characters we get a decomposition of this:

$$
\mathcal{M}=\coprod_{\alpha} \mathcal{M}_{\alpha}=\coprod\{\mathrm{E} \in \operatorname{Coh}(\mathrm{Y}) \mid \operatorname{ch}(\mathrm{E})=\alpha\} .
$$

For example the stack

$$
\mathcal{M}^{\mathcal{O}}=\{(\mathrm{E}, s) \mid \mathrm{E} \in \mathrm{Coh}(\mathrm{Y}), s: \mathcal{O} \rightarrow \mathrm{E}\}
$$

parameterising sheaves together with a section has an obvious forgetful map $\mathcal{M}^{\mathcal{O}} \rightarrow \mathcal{M}$, thus making it into an element $1_{C o h}^{\mathcal{O}}$ of $\mathrm{K}_{0}(\mathrm{~S} t / \mathcal{M})$. Similarly, the Hilbert scheme H ilb is an element of $\mathrm{K}_{0}(\mathrm{~S} t / \mathcal{M})$.

Anyway, coherent sheaves fit into short exact sequences and we want to use this. Let

$$
\mathcal{E}=\left\{0 \rightarrow \mathrm{~A}_{1} \rightarrow \mathrm{~B} \rightarrow \mathrm{~A}_{2} \rightarrow 0\right\}
$$

be the stack parameterising short exact sequences of coherent sheaves. This comes with three maps $a_{1}, b, a_{2}: \mathcal{E} \rightarrow \mathcal{M}$ sending a short exact sequence to the left, middle or right element.

We then stick this in a diagram.

$$
\begin{aligned}
& \mathcal{E} \xrightarrow{b} \mathcal{M} \\
&\left(a_{1}, a_{2}\right) \\
& \mathcal{M} \times \mathcal{M} \\
& \mathrm{H}(\operatorname{Coh}(\mathrm{Y}))=\left(\mathrm{K}_{0}(\mathrm{~S} t / \mathcal{M}), *\right)
\end{aligned}
$$

and the convolution $f * g$ of two elements $f=[\mathrm{V} \rightarrow \mathcal{M}], g=[\mathrm{W} \rightarrow \mathcal{M}]$ is

where $\mathrm{Z}=\mathrm{V} \times \mathrm{W} \times_{\mathcal{M} \times \mathcal{M}} \mathcal{E}$ is the fibre product.

How does this relate to DT invariants? Recall that we think of our generating series $\mathrm{DT}(\mathrm{X})=$ $\sum_{\beta, n} \mathrm{DT}_{\mathrm{X}}(\beta, n) q^{(\beta, n)}$ as living in a ring of formal series R . There is a so-called integration morphism

$$
\mathrm{I}: \mathrm{H}(\mathrm{Coh}(\mathrm{Y})) \rightarrow \mathrm{R}
$$

which, roughly, sends an element $a=\left[\mathrm{V} \rightarrow \mathcal{M}_{(\beta, n)} \rightarrow \mathcal{M}\right]$, for V a scheme, to $\mathrm{I}(a)=\chi_{t o p}(\mathrm{~V}) q^{(\beta, n)}$. Therefore, if $\mathrm{V}=\operatorname{Hilb}_{\leq 1}(\mathrm{X})$ then $\mathrm{I}\left(\mathrm{Hilb}_{\leq 1}\right)=\mathrm{DT}(\mathrm{X})$.

Notice that in the discussion we didn't really have to take $\operatorname{Coh}(\mathrm{Y})$, any other sensible abelian category would have been OK. You might have guessed it, there is a Hall algebra $\mathrm{H}(\mathrm{P} \operatorname{er}(\mathrm{Y} / \mathrm{X}))=$ $\left(\mathrm{K}_{0}(\mathrm{~S} t / \underline{\mathrm{Per}}(\mathrm{Y} / \mathrm{X})), *\right)$ of perverse coherent sheaves. In the algebra we find the usual elements $1_{\mathrm{Per}}, 1_{\mathrm{P} e r}^{\mathcal{O}}$ and $\mathrm{H} i l b_{\mathrm{Per}}$. The algebra also has an integration morphism, so we call

$$
\mathrm{I}\left(\mathrm{H} i l b_{\mathrm{Per}}\right)=\mathrm{DT}(\mathrm{Y} / \mathrm{X})
$$

This quantity is clearly invariant under the flop (the equivalence between $\mathrm{P} \operatorname{er}(\mathrm{Y} / \mathrm{X})$ and $\operatorname{Per}(\mathrm{W} / \mathrm{X})$ sends $\mathcal{O}_{\mathrm{Y}}$ to $\mathcal{O}_{\mathrm{W}}$ and surjections to surjections).

The goal is to find some identity in $\mathrm{H}(\mathrm{Per})$ relating $\mathrm{H} i l b_{\mathrm{Per}}$ with the ordinary $\mathrm{H} i l b$.
We can check directly from the definition of Per , that any $\mathrm{P} \in \mathrm{Per}$ sits in a unique short exact sequence

$$
\mathrm{H}^{-1}(\mathrm{P})[1] \rightarrow \mathrm{P} \rightarrow \mathrm{H}^{0}(\mathrm{P})
$$

Let's call

$$
\begin{gathered}
\mathcal{F}=\left\{\mathrm{F} \in \operatorname{Coh}(\mathrm{Y}) \mid f_{\star} \mathrm{F}=0\right\} \\
\mathcal{T}=\left\{\mathrm{T} \in \operatorname{Coh}(\mathrm{Y}) \mid \mathrm{R}^{1} f_{\star} \mathrm{T}=0\right\}
\end{gathered}
$$

(again, the definition is slightly incomplete - I'm missing an assumption relative to the choice of the perversity). Then

$$
1_{\mathrm{Per}}=1_{\mathcal{F}[1]} * 1_{\mathcal{T}}
$$

in $\mathrm{H}(\mathrm{Per})$. Moreover, one can check

$$
1_{\mathrm{P} e r}^{\mathcal{O}}=1_{\mathcal{F}[1]}^{\mathcal{O}} * 1_{\mathcal{T}}^{\mathcal{O}}
$$

Combining what we know,

$$
\mathrm{H}^{\mathrm{il}} b_{\mathrm{P} e r}=1_{\mathrm{P} e r}^{\mathcal{O}} * 1_{\mathrm{P} e r}^{-1}=1_{\mathcal{F}[1]}^{\mathcal{O}} * 1_{\mathcal{T}}^{\mathcal{O}} * 1_{\mathcal{T}}^{-1} * 1^{-1} \mathcal{F}[1] .
$$

Which is the same as

$$
\mathrm{H} i l b_{\mathrm{P} e r}=1_{\mathcal{F}[1]}^{\mathcal{O}} * 1_{\mathcal{F}[1]}^{-1} * 1_{\mathcal{F}[1]} * 1_{\mathcal{T}}^{\mathcal{O}} * 1_{\mathcal{T}}^{-1} * 1^{-1} \mathcal{F}[1] .
$$

It turns out that after we use the integration morphism for $\mathrm{H}(\mathrm{Per})$ we just need to consider (related to a Poisson structure which is preserved by I)

$$
\left[1 _ { \mathcal { F } [ 1 ] } ^ { \mathcal { O } } * 1 _ { \mathcal { F } [ 1 ] } ^ { - 1 } \left[* \left[1_{\mathcal{T}}^{\mathcal{O}} * 1_{\mathcal{T}}^{-1}[.\right.\right.\right.
$$

A quick computation in $\mathrm{H}(\mathrm{Coh}(\mathrm{Y}))$ reveals the following:

$$
\mathrm{H} i l b=1_{\mathrm{C} o h}^{\mathcal{O}} * 1_{\mathrm{Coh}}^{-1}=1_{\mathrm{T}}^{\mathrm{O}} * 1_{\mathrm{F}}^{\mathcal{O}} * 1_{\mathrm{F}}^{-1} * 1_{\mathcal{T}}^{-1}
$$

However, as $f_{\star} \mathrm{F}=0$ for any $\mathrm{F} \in \mathcal{F}$, we deduce that they do not have any sections and therefore $1_{\mathcal{F}}^{\mathcal{O}}=1_{\mathcal{F}}$, therefore

$$
\mathrm{H} i l b=1_{\mathcal{T}}^{\mathcal{O}} * 1_{\mathcal{T}}^{-1} .
$$

We deduce that in $\mathrm{H}(\mathrm{Per})$

$$
\mathrm{H}_{\mathrm{il}} \mathrm{~b}_{\mathrm{Per}}=\left[1_{\mathcal{F}[1]}^{\mathcal{O}} * 1_{\mathcal{F}[1]}^{-1}[* \mathrm{Hilb} .\right.
$$

Finally, the piece on the left is also related to DT invariants, the final formula is as follows.
Theorem 1.3 - The following series is invariant under flops.

$$
\mathrm{DT}(\mathrm{Y} / \mathrm{X})=\mathrm{DT}_{e x c}^{\vee}(\mathrm{Y}) \mathrm{DT}(\mathrm{Y})
$$

where

$$
\begin{aligned}
\mathrm{DT}(\mathrm{Y}) & =\sum_{\beta, n} \mathrm{DT}_{\mathrm{Y}}(\beta, n) q^{(\beta, n)} \\
\mathrm{DT}_{e x c}(\mathrm{Y}) & =\sum_{\substack{\beta, n \\
f_{\star} \beta=0}} \mathrm{DT}_{\mathrm{Y}}(\beta, n) q^{(\beta, n)} \\
\mathrm{DT}_{e x c}^{\vee}(\mathrm{Y}) & =\sum_{\substack{\beta, n \\
f_{\star} \beta=0}} \mathrm{DT}_{\mathrm{Y}}(\beta, n) q^{(-\beta, n)} .
\end{aligned}
$$

I just want to mention what happens for the McKay correspondence.


Say we have $X$ a CY3 DM stack, with coarse space X . Assume Y to be the irreducible component of the Hilbert scheme of points of $X$ containing the non-stacky points (or in other words the smooth locus of X ). Assume moreover that the fibres of $f$ are at most one-dimensional (this is often called hard Lefschetz, slightly more general than flops). Then what we have is that $\mathrm{D}(\mathcal{X})=\mathrm{D}(\mathrm{Y})$ (this is Bridgeland-King-Reid, later extended by Chen-Tseng). Also, $\mathrm{P} \operatorname{er}(\mathrm{Y} / \mathrm{X})=\mathrm{Coh}(\mathcal{X})$ (I don't think this had yet been written down) so that

$$
\mathrm{DT}(\mathrm{Y} / \mathrm{X})=\mathrm{DT}(X)
$$

Unfortunately, what prevents this from being literally true is that it's not clear whether the two moduli spaces

$$
\begin{array}{r}
\{\mathcal{O} \rightarrow \mathrm{P} \mid \operatorname{ch}(\mathrm{P})=(\beta, n)\} \\
\{\mathcal{O} \rightarrow \mathrm{P} \mid \operatorname{ch}(\mathrm{P})=(\beta, n), \operatorname{dimsupp}(\mathrm{P}) \leq 1\}
\end{array}
$$

are the same (here the epimorphisms are interpreted as being in Per). Since we don't have the lemma that says they are the same $]^{4}$ we need to restrict to the guy on the right, which is open in the guy on the left. In turn this means that we need to take an open subset of Hilb $x$, so that we obtain an identity involving not $\mathrm{DT}(\mathcal{X})$ but invariants defined taking weighted Euler characteristics of some opens in the Hilbert scheme. Nevertheless, the formula formally fits in perfectly with what was predicted (and proved in several non-compact cases) by Bryan-Cadman-Young.

[^2]
[^0]:    ${ }^{1}$ This is a lie - we should be keeping track of different perverse categories, indexed by an integer. We won't do that however.
    ${ }^{2}$ As of now, the theorem is valid for either a quasi-separated scheme or a quasi-compact and separated algebraic space. It's false for stacks though - take two points and $\mathrm{B} \mathbb{Z} / 2 \mathbb{Z}$.

[^1]:    ${ }^{3}$ The extra condition missing in the definition of Per depends on our choice of perversity, which was alluded to when we stated Bridgeland's theorem. The choice of the perversity essentially decides whether $\mathcal{O}_{\mathrm{C}}(-1)[1]$ belongs to Per or $\mathrm{Per}[1]$.

[^2]:    ${ }^{4}$. . . because I wasn't able to prove it.

