A possible introduction to derive categories

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Abstract

This talk is sloppy and let me tell ya: it's not looking good. This is a writeup of some of the things I wanted to talk about. The best place to learn derived categories is of course Daniel Huybrechts's book. In case you were wondering, all the mistakes you find below are mine: you can't have them.

So, the formal aspect of derived and triangulated categories was kicked off by two separate schools: on one side Verdier was trying to massively generalize both Poincaré and Serre duality; on the other side Dold and Puppe were trying to understand the structure of the category of spectra. We might say that the first approach was driven by Homological Algebra, while the second by Homotopy Theory. This was in the '60s. In the '80s Mukai showed why derived categories can be cool from a different perspective, which is what a lot of algebraic geometers these days tend to care about. OK, time for some buzzwords. Derived categories are cool because:

— they provide the natural setting for derived functors and homological algebra;

— Poincaré and Serre duality on steroids;

— stable homotopy theory and spectra (this is more about triangulated/stable categories than derived categories);

— singularity theory, perverse sheaves, D-modules and other words I don't understand;

— homological mirror symmetry, Morita theory, non-commutative geometry, stability conditions and other stuff.

Just to make it clear that derived categories aren't rocket science, let's define one. Let's fix

$$\mathcal{A} = \mathtt{Ab}, \mathrm{R}\text{-}\mathrm{Mod}, \mathtt{Coh}(\mathrm{X}), \ldots$$

(i.e. the category of abelian groups, R-modules, coherent sheaves or any other abelian category). The key feature of $\mathcal{A}$ is the presence of *short exact sequences*

$$0 \to \mathrm{A} \to \mathrm{B} \to \mathrm{C} \to 0$$

i.e. sequences of morphisms where the first is injective, the second is surjective and the image of the first is the kernel of the second. From $\mathcal{A}$ we can form the category $\mathrm{Ch}(\mathcal{A})$ of *chain complexes*, whose objects are strings

$$\cdots \to \mathrm{E}^{p-1} \xrightarrow{d} \mathrm{E}^{p} \xrightarrow{d} \mathrm{E}^{p+1} \to \cdots$$

such that $d \circ d = 0$ and whose morphisms $f\colon \mathrm{E} \to \mathrm{F}$ are given by collections $\{f_p\colon \mathrm{E}^p \to \mathrm{F}^p\}_p$ such that $df = fd$. Two things are important here:

— A chain complex E can be shifted, producing a new complex $\mathrm{E}[1]$ whose p-th object is E^{p+1} and with $d[1] = -d$.

— Every chain complex E has *p-th cohomology*, i.e. the object defined as

$$\mathrm{H}^p(\mathrm{E}) = \frac{\ker\left(\mathrm{E}^p \xrightarrow{d} \mathrm{E}^{p+1}\right)}{\mathrm{im}\left(\mathrm{E}^{p-1} \xrightarrow{d} \mathrm{E}^p\right)}.$$

Any chain morphism f defines a morphism $\mathrm{H}^p(f)$ on the underlying cohomology object. We say that f is a *quasi-isomorphism* if $\mathrm{H}^p(f)$ is an isomorphism for all p.[1] Finally, the *derived category* of $\mathcal{A}$ is

$$\mathrm{D}(\mathcal{A}) := \mathrm{Ch}(\mathcal{A})[\mathrm{qis}^{-1}].$$

The latter category can be defined as having the same objects as $\mathrm{Ch}(\mathcal{A})$ and with morphisms given by formally inverting all quasi-isomorphisms. Whatever, we don't need to obsess with the technical details of all this. What might be worthwhile highlighting is that the key feature of $\mathrm{D}(\mathcal{A})$ is the presence of so-called *exact triangles*

$$\mathrm{A} \to \mathrm{B} \to \mathrm{C} \to \mathrm{A}[1]$$

where $\mathrm{A}[1]$ is just the complex A with a shift in grading. I haven't told you how you recognise that a sequence of morphims as above makes up an exact triangles but for example a short exact sequence in $\mathcal{A}$ induces an exact triangle in $\mathrm{D}(\mathcal{A})$. Taking cohomology of an object of the derived category still makes sense and given an exact triangle as above one has a cohomology long exact sequence (in $\mathcal{A}$):

$$\cdots \mathrm{H}^{p-i}(\mathrm{C}) \to \mathrm{H}^p(\mathrm{A}) \to \mathrm{H}^p(\mathrm{B}) \to \mathrm{H}^p(\mathrm{C}) \to \mathrm{H}^{p+1}(\mathrm{A}) \to \cdots$$

The original category $\mathcal{A}$ sits inside $\mathrm{D}(\mathcal{A})$ fully faitfully:

$$\mathcal{A} \subset \mathrm{D}(\mathcal{A})$$

where an object A is sent to the complex $\{\mathrm{A}^p\}_p$, with $\mathrm{A}^p = 0$ except $\mathrm{A}^0 = \mathrm{A}$. Derived categories aren't abelian, but they are *triangulated* (i.e. there is a distinguished collection of exact triangles). I write this not because I want to tell you what a triangulated category is, but rather that while we've lost some rigidity in structure, there's plenty of structure there (and it's in many ways more flexible, albeit more complicated).

On of the reasons people consider deriving[2] a category is to have a proper place for derived functors to live in. Let's recall a theorem where you might have seen them already.

Theorem (Universal Coefficients) – Let X be a topological space. Let M be an abelian group. Then there are two (non-naturally split) short exact sequences

$$0 \to \mathrm{H}_i(\mathrm{X}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathrm{M} \to \mathrm{H}_i(\mathrm{X}, \mathrm{M}) \to \mathbf{Tor}_1^{\mathbb{Z}}(\mathrm{H}_{i-1}(\mathrm{X}, \mathbb{Z}), \mathrm{M}) \to 0$$
$$0 \to \mathbf{Ext}_{\mathbb{Z}}^1(\mathrm{H}_{i-1}(\mathrm{X}, \mathbb{Z}), \mathrm{M}) \to \mathrm{H}^i(\mathrm{X}, \mathrm{M}) \to \mathbf{Hom}_{\mathbb{Z}}(\mathrm{H}_i(\mathrm{X}, \mathbb{Z}), \mathrm{M}) \to 0. \quad \star$$

The group Tor_1 (and similarly for Ext) is part of a family $\{\mathrm{Tor}_i\}_i$ of groups, called the derived functors of the tensor product. What happens in the derived category is that, for say two R-modules M,N, there is a complex $\mathrm{M} \otimes_{\mathrm{R}}^{\mathrm{L}} \mathrm{N} \in \mathrm{D}(\mathrm{R}\text{-Mod})$ such that $\mathrm{H}^i(\mathrm{M} \otimes^{\mathrm{L}} \mathrm{N}) = \mathrm{Tor}_{-i}^{\mathrm{R}}(\mathrm{M}, \mathrm{N})$. Thus leading to a uniform and unified (and uniwhateverelse) treatment of all your favourite derived functors:

— $\mathrm{Tor}_i(\mathrm{M}, \mathrm{N}) \rightsquigarrow \mathrm{M} \otimes^{\mathrm{L}} \mathrm{N}$

[1] If you're into algebraic topology, you might think of chain complexes always as being complexes of chains (or cochains) on some topological space. So a quasi-isomorphism is like a morphism of spaces inducing an isomorphism on all homology groups.

[2] As far as I know no one's ever taken the second derivative of an abelian category.

— $\mathrm{Ext}^i(A, B) \rightsquigarrow \mathrm{RHom}(A, B)$

— $H^i(X, \mathscr{F}) \rightsquigarrow R\Gamma(X, \mathscr{F})$

— $R^i f_* \mathscr{F} \rightsquigarrow Rf_* \mathscr{F}$

— $L_i f^* \mathscr{F} \rightsquigarrow Lf^* \mathscr{F}$

— $\Omega_X \rightsquigarrow L_X$

— ...

Even better, derived functors at level of derived categories clean up other results which were previously only statable in terms of spectral sequences. A good example is the Leray spectral sequence: $H^p(Y, R^q f_* \mathscr{E}) \Rightarrow H^{p+q}(X, \mathscr{E})$ and more generally[3] $R^p g_*(R^q \mathscr{E}) \Rightarrow R^{p+q}(g \circ f)_* \mathscr{E}$. If we throw a third map in, spelling things out via spectral sequences becomes quite cumbersome. With derived categories it simply becomes: $Rh_* Rg_* Rf_* = R(hgf)_*$.

Remark. The obvious caveat is that while this statement is much neater than one stated in terms of spectral sequences, it doesn't give more information if one is interested in computing the ranks of some specific cohomology groups. At the end of the day (or, rather, at the end of the computation), spectral sequences (or equivalent arguments involving filtrations and the like) cannot be avoided. *

Another place where derived tensor products come up is in intersection theory. Let Y and Z be two subvarieties of, say, projective n-space. Assume that $\dim(Y) + \dim(Z) = n$. When Y and Z intersect transversely, we have the following relation in homology (or Chow groups)

$$[Y] \cdot [Z] = [Y \cap Z]$$

where (for now) here we are viewing $Y \cap Z$ as just a set-theoretic intersection. When the intersection is non-transverse, this breaks down. This has to do with the fact that we were thinking $Y \cap Z$ as $(\mathcal{O}_Y \otimes \mathcal{O}_Z)_{\mathrm{red}}$, which just remembers the *set* $Y \cap Z$. But scheme theory taught us that the intersection $Y \cap Z$ should be thought as a scheme, so we should really be considering $\mathcal{O}_Y \otimes \mathcal{O}_Z$. In simple cases this solves all problems, and the formula still holds. Unfortunately this is not always enough, the correct intersection number at a point p is given not by the dimension of the local ring $\mathcal{O}_{Y,p} \otimes \mathcal{O}_{Z,p}$ but rather by *Serre's intersection formula*

$$\sum_i (-1)^i \dim \mathbf{Tor}_i(\mathcal{O}_{Y,p}, \mathcal{O}_{Z,p}).$$

But what about $[Y \cap Z]$? In turns out that if we interpret the whole derived tensor product $\mathcal{O}_Y \otimes^{L} \mathcal{O}_Z$ as being the ring of functions of a *derived intersection* $Y \cap Z$ then the formula still holds. But let's say a few words about duality theorems.

We have the following important theorem.

THEOREM (Poincaré Duality) – Let X be a smooth, connected and orientable manifold of dimension n. Then there exists an isomorhism

$$H^p(X, \mathbb{Q}) \simeq H_c^{n-p}(X, \mathbb{Q})^\vee$$

between singular cohomology and compactly supported cohomology. *

The generalisation of this by Verdier involves relaxing both the orientable and smoothness assumptions and, more relevantly, extending it to a relative setting.

[3]Taking cohomology is the special case of pushforward where the target is a point.

THEOREM (Verdier Duality) – Let $f\colon X \to Y$ be a map between topological spaces.[4] The compactly supported pushforword $f_!$ has a *right* adjoint at the level of derived categories, namely

$$\mathrm{Hom}(Rf_!\mathscr{F}, \mathscr{G}) \simeq \mathrm{Hom}(\mathscr{F}, Lf^!\mathscr{G})$$

where the Hom spaces are in the derived categories of sheaves of abelian groups. (actually, this works more generally for sheaves of modules over a commutative ring of finite global dimension). $\ast$

Remark. Ordinary duality is recovered by taking Y to be a point. Let's sketch how.

— The first thing to know is that $\mathrm{Hom}(\mathbb{Q}_X, \mathbb{Q}_X[p]) = H^p(X, \mathbb{Q}) = R^p f_* \mathbb{Q}_X$ and $H_c^{n-p}(X, \mathbb{Q}) = R^{n-p} f_! \mathbb{Q}_X$.

— The second thing to know is that, for X smooth and orientable, $Lf^!\mathbb{Q} = \mathbb{Q}_X[n]$. (essentially smoothness guarantees the niceness of the shriek-pullback and orientability makes sure you actually get the constant sheaf and not a local system).

— Consider the special case of Verdier duality with constant rational coefficients:

$$\begin{aligned} & H^p(X, \mathbb{Q}) \simeq \\ \text{(singular cohomology = sheaf cohomology)} &\simeq \mathrm{Hom}(\mathbb{Q}_X, \mathbb{Q}_X[p]) \\ &\simeq \mathrm{Hom}(\mathbb{Q}_X, \mathbb{Q}_X[n+p-n]) \\ \text{(smoothness and orientability)} &\simeq \mathrm{Hom}(\mathbb{Q}_X[n-p], Lf^!\mathbb{Q}) \\ \text{(Verdier duality)} &\simeq \mathrm{Hom}(Rf_!\mathbb{Q}_X[n-p], \mathbb{Q}) \\ \text{(taking duals in the derived category of rational vector spaces)} &\simeq \mathrm{Hom}(\mathbb{Q}_X, Rf_!\mathbb{Q}_X[n-p])^\vee \\ \text{(singular = sheaf compactly supported cohomology)} &\simeq H_c^{n-p}(X, \mathbb{Q})^\vee. \quad \ast \end{aligned}$$

On the level of *coherent* sheaves, then one has an analogous statement.

THEOREM (Serre Duality) – Let X be a smooth and projective variety of dimension n and let $\omega_X = \wedge^n \Omega_X$ be the canonical bundle. Then for any vector bundle $\mathscr{F}$ we have

$$H^i(X, \mathscr{E}) \simeq H^{n-i}(X, \mathscr{E}^\vee \otimes \omega_X). \quad \ast$$

Again, the most relevant generalisation is to the relative setting.

THEOREM (Grothendieck Duality) – Let $f\colon X \to Y$ be a *proper* morphism between noetherian schemes. Then the pushforward functor f_* has a right adjoint on the level of derived categories of coherent sheaves, namely

$$\mathrm{Hom}(Rf_*\mathscr{F}, \mathscr{G}) \simeq \mathrm{Hom}(\mathscr{F}, Lf^!\mathscr{G})$$

where the Hom groups are in the derived category of coherent sheaves. $\ast$

We remark that in the coherent setting, generalising this to non-proper maps is a lot more complicated than in the topological setting, essentially due to the fact that a $f_!$ functor would destroy coherence. Nevertheless, such a generalisation exists. When $Y = \operatorname{Spec} k$ is the ground field and Y is smooth and projective, we have $Lf^! k = \omega_X[n]$ so that we recover Serre duality.

In the topological setting there is one big circle of cool ideas, which doesn't have anything to do with derived categories, but is rather a manifestation of richer triangulated categories. From a topological space X we can extract a chain complex of abelian groups, namely C(X) the complex of singular

[4] Some assumptions are needed, but are incredibly mild.

cochains. As we know from algebraic topology, $H^p(C(X))$ is precisely the p-th singular cohomology group of X. In the homotopy category of spaces we have two things which remind us of derived categories:

— the suspension functor Σ acts as a shift on the level of homology (in the sense that $C(\Sigma X)$ and $C(X)[1]$ are quasi-isomorphic)

— for $X \hookrightarrow Y$ an inclusion of cell complexes we have that the cone Z is homotopy equivalent to Y/X; in turn this gives a short exact sequence of complexes $C(X) \to C(Y) \to C(Z)$ and thus a long exact sequence in cohomology. More generally, for any map $Y \to X$ there is a so-called *cone* $\mathrm{cone}(f)$ (essentially take the cylinder on Y, glue $Y \times \{0\}$ to a point and $Y \times \{1\}$ to X via f), the sequence $Y \to X \to \mathrm{cone}(f)$ should be thought as an exact triangle.

A question to ask might be: is the (homotopy) category of topological spaces in some sense a derived category? This would be nice, as we could interpret taking homology as being a functor between a putative derived category of spaces to $D(\mathtt{Ab})$.

The answer is no, but there are good reasons to try and make it look like one (the famous Brown representability is certainly among those reasons). First off, let's only consider based spaces. This allows the space consisting of a single point to be a *zero* (i.e. initial and final) object of the category of spaces. Of course, we are lacking a fundamental thing: (homotopy classes) of maps between spaces do not form a group. However, there is this neat theorem, which can be interpreted as saying that topological spaces can be "linearised" by suspending enough times.

Theorem – Let X and Y be finite pointed CW complexes. Then, the limit

$$[X, Y]_s = \varinjlim_{n} [\Sigma^n X, \Sigma^n Y]$$

is an abelian group, called the group of *stable* homotopy classes of maps. The letter Σ stands for the (reduced, since we have basepoints) suspension. *

This is great, as for example the *stable homotopy groups* $[S^k, X]_s$ of a space are usually more tractable than the ordinary (or "unstable") ones. Notice that, at the level of cohomology, suspending essentially shifts the complex of singular cochains (in the sense that $C(\Sigma X)$ and $C(X)[1]$ are quasi-isomorphic). At this point though, we would also like to be able to *de-suspend* a space, in other words we would like for Σ to be an equivalence (just as $[1]$ is). As the loop space functor Ω is adjoint to Σ it gives the natural candidate for de-suspension. At the level of spaces the two are *not* inverse to each other, but by loosening up topological spaces and introducing the category of *spectra* one actually achieves precisely this. The (homotopy) category of spectra finally has the same structure of a derived category, i.e. it is a triangulated category. In some sense, spectra should be considered as the analogue of Ab in (stable) homotopy theory. In fact, what people also try to do is to consider "commutative rings" in spectra. For Ab one gets the usual commutative rings and one can does (Grothendieck-style) algebraic geometry. There is so-called *brave new algebraic geometry* where one does algebraic geometry over commutative ring spectra. This is all very cool, but I don't really understand any of it. So let's get on to what this talk is actually supposed to be about.

I decided to start a new section

Let's start from very very very far away. Let X be a smooth manifold, together with its algebra of real-valued smooth functions $C^\infty(X, \mathbb{R})$. Let $\mathrm{MSpec}\, X$ be the set of maximal ideals of this algebra. There is a neat theorem.

Theorem (it's usually attributed to Gelfand and Naimark but it's not theirs in this form) – The set $\mathrm{MSpec}\, X$ has a natural topology and differentiable structure. There is a natural diffeomorphism between X and $\mathrm{MSpec}\, X$. Moreover, there is an equivalence of categories between smooth manifolds and whatever algebras you get by taking Spec. In particular, two smooth manifolds X and Y are *diffeomorphic* if and only if $C^\infty(X) \simeq C^\infty(Y)$ are isomorphic as $\mathbb{R}$-algebras. $\ast$

This theorem represents a starting point for non-commutative geometry, in that one might take all algebras of a particular shape and treat them as functions on a topological space.[5] The idea is that a maximal ideal is of the form $\mathfrak{m}_x$, the ideal of smooth functions vanishing at $x \in X$.

In algebraic geometry this doesn't work. The theorem above only captures manifolds up to *diffeomorphism*. To make things worse, recall the following basic fact. Smooth algebraic varieties (over $\mathbb{C}$) are in particular complex manifolds. In particular again, regular functions on a variety are holomorphic functions. The ring $\mathcal{O}_X$ of holomorphic functions $X \to \mathbb{C}$, for X *compact*, consists only of the constants. Therefore complex manifolds are not just algebras. Of course, for *affine* varieties (zero loci of polynomials in $\mathbb{C}^n$) this does indeed work. This is Hilbert's nullstellensatz (or a tautology, depending on how you think of algebraic geometry).

However, there is a different thing we might do. Given a manifold X we can consider vector bundles on it. Here we have another neat theorem.

Theorem (usually called Serre-Swan, I don't think it's theirs either) – Let X be a smooth manifold. There is an equivalence of categories between $\mathrm{Vect}(X)$ and the category of finitely generated projective modules over $C^\infty(X)$. The equivalence goes by taking $E \to X$ to the set of (smooth) global sections $\Gamma(X, E)$. $\ast$

Right, on the algebraic side we can run a similar story. We have algebraic vector bundles $E \to X$. Again these form a category $\mathrm{Vect}(X)$. For categorical/homological reasons it's a good idea to enlarge this category to that of $\mathtt{Coh}(X)$, coherent sheaves on X. Basically, in general you cannot form the cokernel of a map between vector bundles and so we want to enlarge the category so that we now can: $\mathtt{Coh}(X)$ is an abelian category.[6] We can think of passing from vector bundles to coherent sheaves as passing from projective modules to (finitely generated) modules. On a ring this is literally true, literally. Anyway there is this neat little theorem.

Theorem (Gabriel) – Let X and Y be two varieties.[7] Then X is isomorphic to Y if and only if $\mathtt{Coh}(X)$ is equivalent to $\mathtt{Coh}(Y)$. $\ast$

So the category of coherent sheaves is rich enough to remember X entirely.

Remark. For experts. Gabriel's theorem is false for algebraic stacks. Take $B\mathbb{Z}/2\mathbb{Z}$ and the disjoint union of two points. If one considers $\mathrm{Coh}(X)$ as a *monoidal* category (that is, a category equipped with a notion of *tensor products*) then there is a *Tannakian* version of Gabriel's theorem to stacks. *

OK, but what about derived categories? Well, we know how to define the derived category $D(X) := D(\mathtt{Coh}(X))$ and we know it's good for derived functors, but what else is it nice for? It turns out that Gabriel's theorem breaks down at the level of derived categories (for most varieties anyway). A good way to remain optimistic is to think about trying to classify all (smooth, say) varieties. As it's really hard to do, one might try to relax the problem a bit and try to classify varieties up to *birational* equivalence.[8] Similarly, one might try to bend the classification problem in another direction and identify two varieties if they have the same derived category. Let's see what one can say (the stuff that follows is essentially due many people like Mukai, Bondal, Orlov, Kawamata and others).

First things first, all functors between derived categories are of the following form[9]

[5]There is a peculiar book called *Smooth Manifolds and Observables* by Nestruev which explains this stuff.

[6]An abelian category is a category where we can add morphisms together, take kernels, cokernels and images, all while pretending to be in the category of abelian groups.

[7]The theorem is much more general than that. There are some names which should be mentioned. Insert shameless plug.

[8]Two varieties X and Y are *birational* if and only if they share a common open subset. That is, there exist $U \subset X$, $V \subset Y$ and an isomorphism $U \cong V$. Algebraic geometry is rigid enough that this notion proves to be very interesting. Ah, it might be useful to mention that on a variety any Zariski open is dense.

[9]Some people might try to convince you that away from smooth and projective varieties this might no longer hold. Just cover your ears and do not listen.

Theorem – Let X, Y be smooth and projective varieties. Then for any functor $F\colon D(X) \to D(Y)$ there exists an object $E \in D(X \times Y)$ such that $F = Rq_*(E \otimes^L p^*(-))$, where $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ are the two projections. *

Let's make a big list.

Theorem – Let X and Y be smooth and projective varieties. Then if $D(X) \simeq D(Y)$ then[10]

- $\dim(X) = \dim(Y)$
- $\chi_{\mathrm{top}}(X) = \chi(Y)$
- $H^{\mathrm{odd}}(X, \mathbb{Q}) \simeq H^{\mathrm{odd}}(Y, \mathbb{Q})$ and $H^{\mathrm{even}}(X, \mathbb{Q}) \simeq H^{\mathrm{even}}(Y, \mathbb{Q})$ (the grading is *not* preserved)
- $\sum_p h^{p,p-i}(X) = \sum_p h^{p,p-i}(Y)$
- $\sum_p h^{p-i+n,p}(X) = \sum_p h^{p-i+n,p}(Y)$
- $\mathrm{kod}(X) = \mathrm{kod}(Y)$ but actually,
- the whole (anti-)canonical ring is preserved: $R(X, \omega_X^{\pm 1}) \simeq R(Y, \omega_Y^{\pm 1})$
- $\omega_X^{\pm 1}$ is nef if and only if $\omega_Y^{\pm 1}$ is nef
- $\mathrm{ord}(\omega_X) = \mathrm{ord}(\omega_Y)$ in Pic (e.g. if $\omega_X = \mathcal{O}_X$ then $\omega_Y = \mathcal{O}_Y$)
- if $\omega_X^{\pm 1}$ is ample then $X \simeq Y$
- if $\dim X = 1$ then $X \simeq Y$
- if $\mathrm{kod}(X, \omega_X^{\pm 1}) = \dim(X)$ then X and Y are K-equivalent (and in particular birational)
- (Popa-Schnell) $h^{1,0}(X) = h^{1,0}(Y)$ and $h^0(X, T_X) = h^0(Y, T_Y)$
- if $\dim X = 3$ then $h^{p,q}(X) = h^{p,q}(Y)$.

However, even for threefolds $\pi_1(X)$ is not a derived invariant. *

I'll recall what the Kodaira dimension is, in case you've forgotten.

Definition – Let $L \in \mathrm{Pic}(X)$. The *Kodaira dimension* $\mathrm{kod}(X, L)$ is the integer m such that $h^0(X, L^l) = \dim H^0(X, L^l)$ (called the *plurigenus*, when $L = \omega_X$) grows like a degree m polynomial (for $l \gg 0$). We say $\mathrm{kod}(X, L) = -\infty$ if $h^0(X, L^l) = 0$ for $l > 0$. Equivalently (for $\mathrm{kod}(X, L) \geq 0$)

$$\begin{aligned} \mathrm{kod}(X, L) &= \max \left\{ \dim(\mathrm{im}\varphi_{L^l}) \,\middle|\, l \geq 0 \right\} \\ &= \mathrm{trdeg}_{\mathbb{C}} \mathrm{Frac}(R(X, L)) - 1. \end{aligned}$$

Here $\varphi_L\colon X \dashrightarrow \mathbb{P}^{h^0(L)-1}$ is the rational map induced by the linear system of the line bundle $|L|$; while

$$R(X, L) = \bigoplus_{l \geq 0} H^0(X, L^l)$$

is the *canonical ring* of L. In particular, from the second description we have $\mathrm{kod}(X, L) \leq \dim(X)$. When ω_X one calls $\mathrm{kod}(X) = \mathrm{kod}(X, \omega_X)$ the *Kodaira dimension of* X and $R(X) = R(X, \omega_X)$ the canonical ring. *

[10] I'll write $\omega_X^{\pm 1}$ to mean either ω_X or its dual $\omega_X^\vee$.

I would like to conclude by saying two words on the minimal model program. For derived (and more generally triangulated) categories there is a notion of *semiorthogonal decomposition*.

Proposition – We say that $D(X)$ is *decomposable* if there exist two non-trivial triangulated subcategories D_1, D_2 such that

— for any $E \in D(X)$ there exists a triangle $E_1 \to E \to E_2 \to$ with $E_i \in D_i$

— $\mathrm{Hom}(D_i, D_j) = 0$ for $i \neq j$.

Then $D(X)$ is decomposable if and only if X is disconnected. *

The second condition in the proposition very strong and we say that the D_i are completely orthogonal. If we require the Hom's to vanish only in one direction, $\mathrm{Hom}(D_2, D_1) = 0$ then we say that D_1 and D_2 are *semiorthogonal*. The first condition above and semirthogonality should be interpreted as breaking $D(X)$ into smaller, but not disjoint, pieces.

Theorem (Beilinson) – For projective space $\mathbb{P}^n$ we have

$$D(\mathbb{P}^n) = \langle \mathcal{O}, \ldots, \mathcal{O}(n) \rangle$$

in the sense that there is a semiorthogonal decomposition where each piece is the smallest category generated by the oject $\mathcal{O}(i)$ (which in turn is isomorphic to the derived category of vector spaces). *

This says that $D(\mathbb{P}^n)$ is actually quite simple. What about other results?

Remark (Bridgeland). If $\omega_X = \mathcal{O}_X$ then $D(X)$ does not admit any non-trivial semiorthogonal decompositions. *

Theorem (Bondal-Orlov) – If $Y \to X$ is a blow up with smooth centre then there is a semiorthogonol decomposition

$$D(Y) = \langle D(X), \text{ other stuff having to do with the exceptional locus } \rangle$$

If $Y \to X \leftarrow W$ is a standard flop then $D(Y) \simeq D(W)$. If $Y \to X \leftarrow W$ is a standard flip then we have a semiorthogonol decomposition

$$D(Y) = \langle D(W), \text{ other crap } \rangle$$

*

In light of this (and iterating the process), it has been speculated that the minimal model programme for X might then be viewed as passing to a sort of "minimal" subcategory of $D(X)$. However, difficulties arise even before bumping into singular varieties (for which a lot of what we know breaks down).

Theorem (Bridgeland) – For general threefold flops $D(Y) \simeq D(W)$, and actually W can be constructed as a moduli space of "perverse pointlike objects" in $D(X)$. *

In higher dimension however much is still open.

For surfaces, Toda has interpreted MMP (so iterations of Castelnuovo's theorem) in terms of Bridgeland stability conditions. To $D(X)$ on attaches $\mathrm{Stab}(X)$, the space (it's a manifold) of stability conditions on X. If $X \to Y$ is a birational morphism of surfaces he proves the existence of a whole open $U(Y) \subset \mathrm{Stab}(X)$ such that for any stability condition $\sigma \in U(Y)$ the moduli space M_σ of σ-stable objects, is isomorphic to Y. If we have a factorisation $X \to Y' \to Y$, where $Y' \to Y$ is the blow up of a point then he shows that $\overline{U(Y)} \cap \overline{U(Y')} \neq \varnothing$. So, for a sequence of contractions of (-1)-curves $X = X_0 \to X_1 \to \cdots \to X_n$ one has opens $U(X_i)$, with intersecting closures, and the passage from X to X_n is seen as a one-parameter family of stability conditions $\{\sigma_t\}$ for $t \in [0, n]$. Hope this makes some sense.