# MATH 465 - FALL 2017

### JOHN CALABRESE

These are my own notes for the first few lectures, use at own risk. For this course we will follow Hall's book [Hall5], which only deals with matrix groups. I wanted to spend some time with general Lie groups, if only to demystify the abstract theory. This document is essentially a condensed version of [War83, Chapter 3] (with plenty of omissions but also a few additions).

## Contents

1. Introduction	1
2. The basics	2
2.1. Lie groups	3
2.2. Lie algebras	3
2.3. Vector fields	3
2.4. Tangent vectors	3
2.5. Related	4
2.6. Lie algebra of a Lie group	4
2.7. Subsets	6
2.8. Submanifolds	6
2.9. Frobenius	6
2.10. Subgroups	7
2.11. Covering spaces	8
2.12. Simply connected Lie groups	9
2.13. Exp	9
2.14. The General Linear Group	10
2.15. Ad	13
2.16. Examples	17
2.17. Homogeneous spaces	18
2.18. More about the examples	20
2.19. Loose ends	21
3. Outroduction	21
References	21

# 1. Introduction

A Lie group is both a group and a manifold. What will we be interested in?

- Examples (matrix groups: closed subgroups of GL).
- Actions  $G\times M\to M$  and quotients M/G.
- Morphisms between Lie groups  $G \rightarrow H$ .

Date: September 19, 2017.

• The special case  $G \to GL(V)$ , i.e. representations.

Let  $\phi: \mathbf{G} \to \mathbf{H}$  be a morphism (i.e. a smooth map respecting the group structure). How can we study it? As usual, a good idea is always to take a linear approximation:  $d\phi$ . We also have an obvious point to pick: the neutral element e. Since  $\phi(e) = e$ , we have a linear map  $\psi = d\phi(e): \mathbf{T}_e \mathbf{G} \to \mathbf{T}_e \mathbf{H}$ , between tangent spaces. So studying those  $\psi$ arising as  $d\phi(e)$  is part of studying Lie groups. Does  $\psi$  satisfy any special properties?

We should somehow incorporate the fact that  $\phi$  was a group homomorphism. For example we could take the left translation  $L^x : G \to G$  taking  $y \mapsto xy$ . We certainly have  $\phi \circ L^x = L^{\phi(x)} \circ \phi$ . However, when passing to the derivative this will change tangent spaces. Why not use conjugation? For  $x \in G$ , we have an automorphism  $C_x : G \to G$ sending y to  $xyx^{-1}$ . Certainly  $\phi \circ C_x = C_{\phi(x)} \circ \phi$ .

The derivative of  $C_x$  is a map  $dC_x(e): T_eG \to T_eG$ . As x varies, we obtain a map (traditionally called the *adjoint representation*) Ad:  $G \to GL(T_eG)$ , sending x to  $dC_x(e)$ . When G is a matrix group, this truly sends a matrix A to conjugation by A.

The chain rule implies then that, for any  $x \in G$ ,

$$\psi \circ \operatorname{Ad}_x = \operatorname{Ad}_{\phi(x)} \circ \psi.$$

This is not quite what we want yet, since the formula we obtained still involves  $\phi$ . To remedy, we take a second derivative. (shocking!) Let  $ad: T_eG \to End(T_eG)$  be the derivative of Ad at e. Notice we identified the tangent space  $T_I GL(T_eG)$  with  $End(T_eG)$ : this is standard, since GL is an open subset of End.

Each tangent vector  $X \in T_eG$ , is sent to  $ad_X \in End(T_eG)$ . In turn, this sends a tangent vector  $Y \in T_eG$  to  $ad_X(Y) \in T_eG$ . It is customary to write

$$ad_X(Y) = [X, Y]$$

and call it the *Lie bracket*. One typically writes  $\mathfrak{g} = T_e G$  and refers to the pair  $(\mathfrak{g}, [,])$  as a *Lie algebra*. Again, in the case of a matrix group G, the Lie algebra  $\mathfrak{g}$  also has a description in terms of matrices and the bracket becomes [X, Y] = XY - YX.

Since  $\phi$  was a group homomorphism, it follows that

$$\psi([\mathbf{X}, \mathbf{Y}]) = [\psi(\mathbf{X}), \psi(\mathbf{Y})].$$

The upshot is that we have a way to "linearize" groups, passing from G to  $\mathfrak{g}$  and from  $\phi$  to  $\psi$ . Finally, to provide a bridge back from algebras to groups, we have the exponential map. This is a smooth map exp:  $\mathfrak{g} \to G$  satisfying the following:

- if  $\phi: \mathbf{G} \to \mathbf{H}$  is a morphism of Lie groups, then  $\exp \circ d\phi(e) = d\phi(e) \circ \exp$
- $\exp(0) = e$
- $d \exp(0) \colon T_0 \mathfrak{g} = \mathfrak{g} \to T_e G = \mathfrak{g}$  is the identity
- lines through the origin are sent to one-parameter subgroups of G.

A one-parameter subgroup is simply a morphism  $\mathbf{R} \to \mathbf{G}$ . In the case of a matrix group, the exponential map is given by the exponential of a matrix  $\mathbf{X} \mapsto e^{\mathbf{X}}$ .

#### 2. The basics

In this section we want to cover the first definitions and recall some basic facts about smooth manifolds.

2.1. Lie groups. For us, a *Lie group* will be a group object in the category of (smooth) manifolds. For us, a manifold will be always be smooth (i.e.  $\mathscr{C}^{\infty}$ ), second countable and Hausdorff (in particular paracompact and admitting partitions of unity). So, a Lie group is a manifold G, together with an element  $e \in G$ , two smooth maps  $m: G \times G \to G$ ,  $i: G \to G$ , satisfying the group axioms. A morphism of Lie groups is a smooth map  $G \rightarrow H$  which is also a group homomorphism (we will also say *Lie map* or *Lie morphism*).

2.2. Lie algebras. Here is one way Lie algebras come up. Let A be an associative algebra. We define the *(commutator)* bracket on A to be  $[,]: A \times A \rightarrow A$  sending x, y to

$$[x, y] = xy - yx.$$

The bracket has the following properties:

- [,] is bilinear
- [x, x] = 0
- [x, y] = -[y, x]
- [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. (the Jacobi identity)

A Lie algebra is a vector space  $\mathfrak{g}$  with a bracket [,] satisfying the axioms above. A morphism of Lie algebras  $\psi: \mathfrak{g} \to \mathfrak{h}$  is a map of vector spaces which preserves the bracket:  $[\psi(x), \psi(y)] = \psi([x, y])$  (we will also say a *Lie map* or *morphism*).

Remark 2.1. Taking an associative algebra A to the corresponding Lie algebra under the commutator bracket actually describes a functor Ass  $\rightarrow$  Lie. There is also a way to go back (a left adjoint), a procedure which takes a Lie algebra g and spits out an associative algebra  $U(\mathfrak{g})$  (called the universal enveloping algebra). We'll not go into that now. We only mention that the two gadgets are closely related: Rep(g), the category of representations of  $\mathfrak{g}$ , is actually equivalent to  $Mod(U(\mathfrak{g}))$ , the category of  $U(\mathfrak{g})$ -modules.

2.3. Vector fields. A vector field X on a smooth manifold M is by definition a selfderivation of the R-algebra of smooth functions  $\mathscr{C}^{\infty}(M)$ . This means that  $X \colon \mathscr{C}^{\infty}(M) \to \mathbb{C}^{\infty}(M)$  $\mathscr{C}^{\infty}(\mathbf{M})$  eats up a function  $f \in \mathscr{C}^{\infty}(\mathbf{M})$  and spits out another function  $\mathbf{X}f$ , which should be thought of as the "derivative of f along the direction X". All this while satisfying the following properties:

- X(af + g) = aXf + Xg for any  $a \in \mathbf{R}, f, g \in \mathscr{C}^{\infty}(M)$
- X(fg) = fXg + gXf for any  $f, g \in \mathscr{C}^{\infty}(M)$  (Leibniz rule)

We write  $\mathfrak{X}(M)$  for the set of vector fields. This is a vector space, but it's not an algebra: composition of vector fields is not a vector field. However, we may pretend it's an algebra (or view it as sitting inside  $\operatorname{End}_{\mathbf{R}}(\mathscr{C}^{\infty}(M))$ , which is an algebra) and define the corresponding bracket:  $[X, Y] = X \circ Y - Y \circ X$ . More explicitly,

$$[\mathbf{X},\mathbf{Y}](f) = \mathbf{X}(\mathbf{Y}f) - \mathbf{Y}(\mathbf{X}f)$$

for any  $f \in \mathscr{C}^{\infty}(M)$ . With this bracket,  $\mathfrak{X}(M)$  becomes a Lie algebra.

2.4. Tangent vectors. If  $p \in M$ , a tangent vector v at p is a derivation-at-p of  $\mathscr{C}^{\infty}(M)$ . This means  $v: \mathscr{C}^{\infty}(M) \to \mathbf{R}$  is such that

- v(af + g) = av(f) + v(g) for all  $a \in \mathbf{R}, f, g \in \mathscr{C}^{\infty}_{\mathbf{M}, p}$  v(fg) = f(p)v(g) + g(p)v(f) for all  $f, g \in \mathscr{C}^{\infty}_{\mathbf{M}, p}$ .

We think of v(f) as being the value at p of the derivative of f along the vector v. We write  $T_pM$  for the set of tangent vectors at p. This is a vector space of dimension dim M.

There is a "restriction to the fibre" or "evaluation" map  $\mathfrak{X}(M) \to T_p M$  which takes a vector field X to the tangent vector  $X_p$ . The latter is defined by  $X_p(f) = (Xf)(p)$ . We may think of a vector field X as a (smoothly varying) collection of tangent vectors  $\{X_p\}_{p\in M}$  (in fancier terms: a section of the tangent bundle). If  $f \in \mathscr{C}^{\infty}(M)$ , we think of (Xf)(p) as the derivative of f at p along the vector  $X_p$ . Notice the difference: vector fields are derivations  $\mathscr{C}^{\infty}(M) \to \mathscr{C}^{\infty}(M)$ , while tangent vectors are derivations  $\mathscr{C}^{\infty}(M) \to \mathbb{R}$ .

If  $\phi \colon M \to N$  is a smooth map and  $\phi(p) = q$ , the *differential at* p (or *total derivative*) is  $d\phi(p) \colon T_p M \to T_q N$ , where the tangent vector v is sent  $d\phi(p)v$  which on a function f evaluates to

$$\mathrm{d}\phi(p)v(f) = v(f \circ \phi).$$

The derivative is **R**-linear. When enough is understood, we write  $\phi_*$  for  $d\phi(p)$ .

**Remark 2.2.** Notice that there isn't a pushforward of vector fields, i.e.  $d\phi$  does not define a map  $\mathfrak{X}(M) \to \mathfrak{X}(N)$ . Indeed, if  $g \in \mathscr{C}^{\infty}(N)$ , you'd want to define  $(\phi_*X)(g)$  as  $X(g \circ \phi)$ . But the latter is a function on M, not on N.

**Remark 2.3.** Let c(t) be a curve in a manifold M. Then  $X = c'(0) = dc(0)1 \in T_x M$  is a tangent vector, where x = c(0). We say X is the *velocity* of c(t) at t = 0. If  $f : M \to \mathbb{R}$  is a function, then

$$\mathbf{X}f = \frac{d}{dt}\Big|_{t=0} f(c(t)) = \lim_{t \to 0} \frac{f(c(t)) - x}{t}.$$

Finally, let  $\phi \colon \mathcal{M} \to \mathcal{N}$  be a smooth map. Then  $\phi_* \mathcal{X} = d\phi(x)\mathcal{X} = (\phi \circ c)'(0)$ .

2.5. Related. Let  $\phi \colon M \to N$  be a map. The vector fields  $X \in \mathfrak{X}(M)$ ,  $Y \in \mathfrak{X}(N)$  are  $\phi$ -related if

$$\mathcal{X}(f \circ \phi) = (\mathcal{Y}f) \circ \phi.$$

Equivalently, for all  $p \in M$  we have  $\phi_* X_p = d\phi(p) X_p = Y_{\phi(p)}$ .

**Lemma 2.4.** Suppose  $\phi \colon M \to N$ ,  $X_1, X_2 \in \mathfrak{X}(M)$ ,  $Y_1, Y_2 \in \mathfrak{X}(N)$ ,  $Y_i$  is  $\phi$ -related to  $X_i$ . Then  $[X_1, X_2]$  is  $\phi$ -related to  $[Y_1, Y_2]$ .

*Proof.* Let  $f \in \mathscr{C}^{\infty}(\mathbf{N})$ . Then

$$\begin{split} [\mathbf{X}_1, \mathbf{X}_2](f \circ \phi) &= \mathbf{X}_1(\mathbf{X}_2(f \circ \phi)) - \mathbf{X}_2(\mathbf{X}_1(f \circ \phi)) \\ &= \mathbf{X}_1(\mathbf{Y}_2f \circ \phi) - \mathbf{X}_2(\mathbf{Y}_1f \circ \phi) \\ &= \mathbf{Y}_1(\mathbf{Y}_2f) \circ \phi - \mathbf{Y}_2(\mathbf{Y}_1f) \circ \phi \\ &= [\mathbf{Y}_1, \mathbf{Y}_2]f \circ \phi. \end{split}$$

2.6. Lie algebra of a Lie group. Let now G be a Lie group. If  $x \in G$ , we have a diffeomorphism  $L^x : G \to G$  sending  $y \mapsto L^x(y) = xy$ . This is called the *left translation*. We say that a vector field  $X \in \mathfrak{X}(G)$  is *left invariant* if for all  $x \in G$  and  $f \in \mathscr{C}^{\infty}(G)$  we have

(2.1) 
$$X(f \circ L^x) = (Xf) \circ L^x$$

In other words, X is  $L^x$ -related to itself, for all  $x \in G$ . Write  $\mathfrak{X}(G)^L \subset \mathfrak{X}(G)$  for the subset of left-invariant vector fields on G.

**Proposition 2.5.**  $\mathfrak{X}(G)^L$  is a Lie subalgebra of  $\mathfrak{X}(G)$ .

A subalgebra is just a vector subspace closed under brackets.

*Proof.* Clearly  $\mathfrak{X}(G)^L$  is a subspace, so it remains to show it is closed under Lie brackets. Suppose  $X, Y \in \mathfrak{X}(G)$  are left invariant. Then both X and Y are  $L^x$ -related to themselves, hence [X, Y] is  $L^x$  related to itself, for all  $x \in G$ .

**Proposition 2.6.** Restriction  $\mathfrak{X}(G)^{L} \to T_{e}G$  is an isomorphism of vector spaces.

*Proof.* Let us construct an inverse to restriction, by forcing left invariance. If  $v \in T_e G$  is a tangent vector at the identity, we wish to define a vector field  $X \in \mathfrak{X}(G)^L$ . In other words, we need to declare the value Xf(x) for any function  $f \in \mathscr{C}^{\infty}(G)$  and point  $x \in G$ . So we set

$$Xf(x) = v(f \circ L^x)$$

this makes sense as we want left invariance:

$$v(f \circ \mathbf{L}^x) = \mathbf{X}(f \circ \mathbf{L}^x)(e) = (\mathbf{X}f \circ \mathbf{L}^x)(e) = \mathbf{X}f(x).$$

Then we go and diligently check that everything we did made sense.

We call  $\mathfrak{g} = \text{Lie}(G) = T_e G = \mathfrak{X}(G)^L$  the *Lie algebra* of the Lie group G. By the proposition above, it is a finite dimensional vector space of dimension dim G.

**Remark 2.7.** Using the previous proposition,  $T_eG$  inherits the structure of a Lie algebra. Although from what we've said so far it doesn't seem natural to view  $T_eG$  as a Lie algebra, it'll pay off when dealing with matrix groups.

If  $X \in T_eG$ , the corresponding vector field  $X \in \mathfrak{X}(G)^L$  is determined by  $X_x = L_*^x X$ .

**Remark 2.8.** What if we had used right invariant vector fields instead? Certainly, restriction  $\mathfrak{X}(G)^R \to T_e G$  still gives us a vector space isomorphism. What changes is that [X, Y] becomes -[X, Y]. This can be shown by using the derivative of the inversion map.

**Proposition 2.9.** Let  $\phi: G \to H$  be a map of Lie groups. Then the derivative  $\phi_*: T_e G \to T_e H$  is a map of Lie algebras.

In other words we have a functor LieGrp  $\rightarrow$  LieAlg sending G to g and  $\phi$  to  $\phi_* = d\phi(e)$ .

*Proof.* We need to check that  $\phi_*$  preserves the Lie bracket. First off, we notice that since  $\phi$  is a group homomorphism,  $\phi \circ L^x = L^{\phi(x)} \circ \phi$ .

Suppose now  $X \in T_e G$  and write  $\tilde{X}$  for the corresponding left invariant vector field on G. Let  $Y = \phi_* X$  and let  $\tilde{Y}$  be the corresponding left invariant vector field on H. The vector field  $\tilde{X}$  and  $\tilde{Y}$  are  $\phi$ -related. Indeed, if  $f \in \mathscr{C}^{\infty}(H)$ , then

$$\begin{split} \tilde{\mathbf{X}}(f \circ \phi)(x) &= \mathbf{X}(f \circ \phi \circ \mathbf{L}^x) \\ &= \mathbf{X}(f \circ \mathbf{L}^{\phi(x)} \circ \phi) \\ &= (\phi_* \mathbf{X})(f \circ \mathbf{L}^{\phi(x)}) \\ &= \mathbf{Y}(f \circ \mathbf{L}^{\phi(x)}) \\ &= \tilde{\mathbf{Y}}f(\phi(x)). \end{split}$$

 $\square$ 

This lets us easily conclude.

$$\begin{aligned} [\phi_* \mathbf{X}_1, \phi_* \mathbf{X}_2] &= [\phi_* \mathbf{X}_1, \phi_* \mathbf{X}_2]_e \\ &= [\tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2]_e \\ (\text{Lemma 2.4}) &= \phi_* [\tilde{\mathbf{X}}_1, \tilde{\mathbf{X}}_2]_e \\ &= \phi_* [\mathbf{X}_1, \mathbf{X}_2] \end{aligned}$$

where  $X_i \in \mathfrak{g}, Y_i = \phi_* X_i$  and  $\tilde{\phantom{a}}$  denotes the associated left invariant vector fields.  $\Box$ 

2.7. Subsets. The following may be viewed as categorical pedantry, but it pays off when stating some theorems. Let  $S = \{1, 2, 3\}$  be a set. No one would argue that  $T = \{1, 2\}$  is a subset of S. But what about  $T' = \{1, \clubsuit\}$ ? There is an obvious injective map  $j: T' \to S$  sending  $1 \mapsto 1$  and  $\clubsuit \mapsto 2$ . Hence we may view T' (together with the map j) as a subset of S. But should we? Well, maybe we should. Define a categorical subset of S to be a pair (j, A) where  $j: A \to S$  is injective. We then declare two subsets (j, A), (j', A') to be equivalent if there is a bijection  $f: A \to A'$  such that  $j' \circ f = j$ . Obviously, given a subset (j, A) we can always find a unique representative in its equivalence class (i, T) where  $T \subset S$  and i is the inclusion.

Now, for sets this line of thought isn't too helpful. However, already in topology we can think of interesting injective maps  $f: X \to Y$ , but where the topology of X is different from the subspace topology of  $f(X) \subset Y$ . Think of exp:  $[0, 2\pi) \to S^1$ : while we could just say that we are viewing  $S^1$  with a funky topology on it, it's just clearer to think about  $[0, 2\pi)$  wrapping around the circle.

A similar circle (pun not intended) of ideas applies to cartesian products (which is possibly the example one encounters of a 'limit' in category theory). Say we have three sets A, B, C. What is their product? Obviously the answer is  $A \times B \times C$ . But, why shouldn't  $(A \times B) \times C$  or  $A \times (B \times C)$  be regarded as the product? Strictly speaking, these are three different sets: the first consists of symbols (a, b, c), the second ((a, b), c) and the third (a, (b, c)). But all of them are clearly representing the triple product of A, B and C.

2.8. Submanifolds. Recall that  $\iota: N \to M$  is an *immersion* if, for all  $p \in N$ ,  $d\psi(p): T_pN \to T_{\psi(p)}M$  is injective. An immersion is a *submanifold* if it is injective. A submanifold is an *embedding* if it is a homeomorphism onto its image: in other words the topology on M coincides with the subspace topology of  $\psi(M) \subset N$ .

We say two submanifolds  $(\iota, N), (\iota', N')$  are equivalent if there is a diffeomorphism  $\phi \colon N \to N'$  making the obvious diagram commute:  $\phi \circ \iota = \iota'$ . Among an equivalence class of submanifolds there is always a unique representative which is a subset of the ambient manifold:  $A \subset M$ . However, the topology of A is not necessarily the subspace topology (unless it is an embedding).

2.9. Frobenius. In this subject, a *distribution* V often denotes a smoothly varying collection of subspaces  $\{V_p\}_{p \in M}, V_p \subset T_p M$ . In other words,  $V \subset TM$  is a sub-bundle of the tangent bundle of M. We say V is *involutive* (or *completely integrable*) if, whenever  $X, Y \in \mathfrak{X}(M)$  belong to V, we have [X, Y] also belongs to V. We say a submanifold  $N \subset M$  is an *integral manifold* of V if  $T_p N = V_p$  for all  $p \in N$ . The following basic result is central for many things that will follow.

**Theorem 2.10** (Frobenius). Let  $V \subset TM$  be an involutive distribution of rank k. Let  $p \in M$ . Then there exists an integral submanifold  $N \subset M$  passing through p.

Even better, there exists a unique connected *maximal* integral manifold N passing through p. Every other connected integral manifold is contained in N.

Here's a consequence, which already shows how close Lie groups and Lie algebras are.

**Theorem 2.11.** Let G be a connected Lie group and let  $\phi, \psi \colon G \to H$  be such that  $\phi_* = \psi_* \colon \mathfrak{g} \to \mathfrak{h}$ . Then  $\phi = \psi$ .

**Proof.** The graph  $x \mapsto (x, \phi(x)) \in G \times H$  induces a map of Lie algebras  $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{h}$ . In particular, its image defines a subspace  $V_e \subset \mathfrak{g} \times \mathfrak{h}$ . By left translating as usual, this defines a sub-bundle  $V \subset T(G \times H)$ . Since  $\phi_*$  preserves the Lie bracket, V is involutive. Hence, by Frobenius, there is a unique connected maximal integral manifold. By maximality, this has to coincide with the graph of  $\phi$ . Since  $\phi_* = \psi_*$ , they both define the same sub-bundle. In turn this means that the graphs of  $\phi$  and  $\psi$  are the same, hence  $\phi = \psi$ .

A way to rephrase this is that the functor from connected Lie groups to Lie algebras is *faithful*.

2.10. **Subgroups**. Let G be a Lie group. We will use a flexible definition of subgroup, making a couple of theorems nicer. A *subgroup* will be a pair  $(\iota, H)$  where

- H is a Lie group
- $\iota : H \to G$  is a Lie map
- $(\iota, H)$  is a submanifold (i.e.  $\iota$  is injective and an immersion)

A subgroup is *closed* if  $\iota(H) \subset G$  is closed. As usual, we place an equivalence relation on subgroups. We will be more interested in closed subgroups, but there are plenty of non closed ones out there. A silly example is to take  $\mathbf{Q}$  with the discrete topology and the inclusion  $\mathbf{Q} \subset \mathbf{R}$ .

A better example is the infamous irrational torus. Consider the map  $\phi: \mathbb{Z} \to S^1$ sending k to  $e^{2\pi i k a}$  for  $a \in \mathbb{R}$ . If  $a \in \mathbb{Q}$ , then the image of  $\phi$  consists of finitely many points. However, if  $a \in \mathbb{R} \setminus \mathbb{Q}$ , then its image is dense. This is because any real number can be approximated by some ka, modulo 1. To obtain the irrational torus, we map  $\psi: \mathbb{R} \to S^1 \times S^1$  sending  $t \mapsto (e^{it}, e^{iat})$  with a again irrational. The image of  $\psi$  is a subgroup of the torus, whose closure is the whole torus.

Here is a general fact which is interesting in its own right.

**Proposition 2.12.** Let G be a topological group. Let  $G^0$  be the connected component containing the identity  $e \in G$ . Then  $G^0$  is a normal subgroup.

**Proof.** Let m denote the multiplication map. The product  $G^0 \times G^0$  is connected and the same will hold for its image  $m(G^0 \times G^0)$ . Since it contains e, we must have  $m(G^0 \times G^0) \subset G^0$ . Same argument for the inversion map  $\iota$ . Hence  $G^0$  is a subgroup. Normality follows similarly as conjugation  $C_q(x) = gxg^{-1}$  is continuous.

Obviously if G is a Lie group,  $G^0$  will be a normal (closed) Lie subgroup. Here is another application of the Frobenius theorem.

**Theorem 2.13.** Let  $\mathfrak{g}$  be the Lie algebra of a Lie group G. Let  $\mathfrak{h} < \mathfrak{g}$  be a subalgebra. Then there is a unique connected Lie subgroup  $(i, \mathrm{H})$  with  $i_*\mathrm{Lie}(\mathrm{H}) = \mathfrak{h}$ .

*Proof sketch.* By left translating,  $\mathfrak{h}$  defines a sub-bundle of TG. A direct computation shows it is involutive. Take  $(i, \mathbf{H})$  to be the unique connected maximal integral manifold passing through e. We want to show that  $\mathbf{H}$  is actually a Lie group. Pick  $x \in i(\mathbf{H}) \subset \mathbf{G}$ .

Then  $(L^{x^{-1}} \circ i, H)$  is also an integral manifold passing through e. By uniqueness, we must have  $L^{x^{-1}}(i(H)) \subset i(H)$ . In other words, for  $y \in i(H)$  we have  $x^{-1}y \in i(H)$ . This implies H can be endowed with the structure of an abstract group, such that  $i: H \to G$  is a group homomorphism. One then checks that everything is smooth and that there is no other Lie group satisfying the theorem.

**Corollary 2.14.** Let G be a Lie group with Lie algebra g. There is a bijection between connected Lie subgroups and Lie subalgebras.

We conclude this subsection by stating without proof a powerful result.

**Theorem 2.15.** Let (i, H) < G be an *abstract* subgroup (i.e. a subgroup in Grp, *not* LieGrp). If H is *closed*, then H is a Lie subgroup and i is an embedding.

*Proof.* A proof can be found in [War83, Th 3.21]. It does not rely on any big sledgehammers, but only the existence of nice coordinate charts for submanifolds and point-set topology.  $\Box$ 

2.11. Covering spaces. Lie groups and covering spaces play nicely.

**Proposition 2.16.** Let G be a connected Lie group. Let  $\pi: \tilde{G} \to G$  be its universal cover. Then  $\tilde{G}$  can be given the structure of a Lie group so that  $\pi$  is a Lie map.

**Proof.** We know that  $\widetilde{G}$  is a manifold and that  $d\pi$  is an isomorphism on tangent spaces. Pick a point in  $\pi^{-1}(e)$  and call it  $\widetilde{e}$ . This will be the neutral element of the group structure on  $\widetilde{G}$ . Consider  $\alpha \colon \widetilde{G} \times \widetilde{G} \to G$  given by  $\alpha(x, y) = \pi(x)\pi(y)^{-1}$ . Since  $\widetilde{G} \times \widetilde{G}$  is simply connected, there exists a unique lift  $\beta \colon \widetilde{G} \times \widetilde{G} \to \widetilde{G}$  sending  $\beta(\widetilde{e}, \widetilde{e}) = \widetilde{e}$ . By lift we mean that  $\pi\beta = \alpha$ . For  $x, y \in \widetilde{G}$ , we then define  $x^{-1} = \beta(\widetilde{e}, x)$  and  $xy = \beta(x, y^{-1})$ . These operations are clearly smooth. One then checks they satisfy the group axioms. By construction,  $\pi$  is a group homomorphism.

Let  $\pi : \widetilde{\mathbf{G}} \to \mathbf{G}$  be our Lie group universal cover. Let  $\mathbf{K} = \ker \pi = \pi^{-1}(e)$  be the kernel. Since  $\pi$  is a covering space, K has the discrete topology. We have  $\mathbf{G} = \widetilde{\mathbf{G}}/\mathbf{K}$  and thus K may be identified with the fundamental group  $\pi_1(\mathbf{G}, e)$ . While we are at it, we mention another general fact about Lie groups.

**Theorem 2.17.** Let G be a connected Lie group:  $\pi_1(G, e)$  is abelian,  $\pi_2(G, e) = 0$  and  $\pi_3(G, e)$  is torsion free.

**Proposition 2.18.** Let  $\phi: G \to H$  be a Lie map. Then  $\phi$  is a covering if and only if  $\phi_*: \mathfrak{g} \to \mathfrak{h}$  is an isomorphism.

*Proof.* Covering spaces are fibre bundles with discrete fibre, hence one direction is obvious. Conversely, suppose  $\phi_*$  is an isomorphism. Then  $d\phi(x)$  is an isomorphism for all x, by translating. This implies  $\phi$  is a local homeomorphism. In particular it's open. It follows that  $\phi$  must be surjective (this relies on the lemma below). Let  $K = \ker \phi$ . Then H = G/K and, since  $\phi$  is a local diffeo, the action of K is a proper discontinuous, hence  $\phi$  is a covering space.

The proof above used the fact that a neighborhood of the identity generates the group.

**Lemma 2.19.** Let G be a Lie group. Let U be a neighborhood of the identity. Then U generates G, in the sense that  $G = \bigcup_n U^n$  where  $U^n$  is the subset of *n*-fold products of elements of U.

#### 2.12. Simply connected Lie groups.

**Theorem 2.20.** Let G, H be Lie groups with Lie algebras  $\mathfrak{g}, \mathfrak{h}$ . Suppose G is simply connected. Let  $\psi : \mathfrak{g} \to \mathfrak{h}$  be a Lie morphism. Then there exists a unique  $\phi : G \to H$  such that  $\phi_* = \psi$ .

*Proof.* Consider the subalgebra of  $\mathfrak{g} \times \mathfrak{h}$  given by  $(x, \psi(x))$  for  $x \in \mathfrak{g}$ . By a previous theorem, we know there is a unique connected Lie subgroup  $Z \subset G \times H$ . Now, the projection  $Z \to G$  induces an isomorphism of Lie algebras, hence it must be a covering space. But G is simply connected, hence  $Z \to G$  is an isomorphism, hence Z must be the graph of some  $\phi \colon G \to H$ , which is forced to be a group homomorphism.  $\Box$ 

A more categorical way to say this is that the functor from simply connected Lie groups to Lie algebras is *full*. Combined with 2.11, we see the functor is actually *fully faithful*. What we would like to say is that the functor is also *essentially surjective*.

**Theorem 2.21** (Ado). Let  $\mathfrak{g}$  be a finite dimensional Lie algebra (over any field). Then there is an injective Lie map  $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$  for some vector space V.

Therefore, let  $\mathfrak{g}$  be any Lie algebra and view it as a matrix Lie algebra. By  $\mathfrak{R}$ , there is a subgroup H < GL(V). Hence, the functor (from simply connected Lie groups to finite dimensional Lie algebras) Lie<sub>s.c.</sub>  $\rightarrow$  LieAlg<sub>fin</sub> is an equivalence.

Finally, we mention a few more interesting facts about Lie groups.

**Theorem 2.22** (people, I should really look up the references). Let G be a second countable, locally euclidean topological group. Then there exists a differentiable structure on G making it a Lie group. This structure is unique. Moreover, this structure can be upgraded (in a unique way) to an analytic structure.

**Theorem 2.23.** Let  $\phi \colon G \to H$  a continuous group homomorphism between Lie groups. Then  $\phi$  is smooth.

2.13. **Exp.** Consider now **R** as a Lie group. The tangent space at 0 has a distinguished basis vector, wich we call d/dr. The Lie algebra of **R** is then the vector space  $\mathbf{R} d/dr$ , with trivial bracket.

Let G be a random Lie group. If  $X\in \mathfrak{g}$  is any element of the Lie algebra, we have a corresponding map

$$(2.2) \qquad \qquad \lambda \, \mathrm{d}/\,\mathrm{d}r \mapsto \lambda \mathrm{X}$$

which is a map of Lie algebras since the bracket on the domain is trivial. Since  $\mathbf{R}$  is simply connected, there is a unique Lie map

$$\exp_{\mathbf{X}} : \mathbf{R} \to \mathbf{G}$$

whose derivative at  $0 \in \mathbf{R}$  is (2.2)

$$d \exp_{\mathbf{X}}(0) \left(\lambda d / dr\right) = \lambda \mathbf{X}.$$

In other words,  $t \mapsto \exp_X(t)$  is the *unique* 1-parameter subgroup of G with that derivative. We define the exponential map to be

$$\exp\colon \mathfrak{g} \to \mathbf{G}$$
$$\mathbf{X} \mapsto \exp_{\mathbf{X}}(1)$$

You'd expect exp to have the following properties.

**Theorem 2.24.** Let  $X \in \mathfrak{g} = \text{Lie}(G)$ .

- (1)  $\exp(t\mathbf{X}) = \exp_{t\mathbf{X}}(1) = \exp_{\mathbf{X}}(t)$ , for all  $t \in \mathbf{R}$
- (2)  $\exp((t_1 + t_2)X) = \exp(t_1X)\exp(t_2X)$ , for all  $t_1, t_2 \in \mathbf{R}$
- (3)  $\exp(-t\mathbf{X}) = \exp(t\mathbf{X})^{-1}$ , for all  $t \in \mathbf{R}$
- (4) exp:  $\mathfrak{g} \to G$  is smooth,

$$d \exp(0) \colon T_0 \mathfrak{g} \to T_e G = \mathfrak{g}$$

is the identity.

- (5) exp induces a diffeomorphism between a neighborhood of  $0 \in \mathfrak{g}$  and a neighborhood of  $e \in G$ .
- (6) The 1-parameter group of diffeomorphisms  $X_t$  associated with the left invariant vector field X is given by  $X_t = R^{\exp_X(t)}$  where  $R^x$  is right-translation by x.

Proof. This is [War83, Thm 3.31].

**Theorem 2.25** (naturality of exp). If  $\phi \colon G \to H$  is a Lie map, the following diagram commutes.

$$\begin{array}{c} \mathfrak{g} \xrightarrow{\phi_*} \mathfrak{h} \\ \exp \downarrow \qquad \qquad \downarrow \exp \\ \mathbf{G} \xrightarrow{\phi} \mathbf{H} \end{array}$$

*Proof.* Let  $X \in \mathfrak{g}$  and let  $Y = \phi_* X$ . By definition,  $\exp(Y) = \exp_Y(1)$ . We know that  $\exp_Y$  is the only 1-parameter subgroup such that  $(\exp_Y)_*$  sends d/dr to Y. On the other hand,  $\phi \circ \exp_X : \mathbb{R} \to \mathbb{H}$  is also a 1-parameter subgroup, and  $(\phi \circ \exp_X)_*$  sends d/dr to  $\phi_* X = Y$ . Hence  $\exp(Y) = \exp_Y(1) = (\phi \circ \exp_X)(1) = \phi(\exp(X))$ .

**Proposition 2.26.** Let (H, i) be a Lie subgroup of G and let  $X \in \mathfrak{g}$ . If  $X \in i_*\mathfrak{h}$ , then  $\exp(tX) \in i(H)$  for all  $t \in \mathbf{R}$ . Conversely, if  $\exp(tX) \in i(H)$  for t in some interval, then  $X \in i_*\mathfrak{h}$ .

*Proof.* One direction follows from naturality of exp, the other from Frobenius (uniqueness of maximal integral manifolds).  $\Box$ 

Finally, a theorem which will be useful to construct Lie groups.

**Theorem 2.27.** Let  $\phi: G \to H$  be a Lie map. Let  $K = \ker \phi$  and  $\mathfrak{k} = \ker \phi_*$ . Then K is a closed Lie group with Lie algebra  $\mathfrak{k}$ .

*Proof.* Being a kernel, A is a closed subgroup and hence (by the closed subgroup theorem) it is a Lie group. We need to show that  $\mathfrak{k}$  is its Lie algebra. Let  $X \in \mathfrak{g}$ . We know by a previous theorem that  $X \in \text{Lie}(K)$  if and only if  $\exp tX \in K$  for all  $t \in \mathbf{R}$ . Now,  $\exp tX \in K$  if and only if  $\phi(\exp tX) = e$  if and only if  $\exp \phi_* tX = e$ . But if  $\exp tY = e$  for all  $t \in \mathbf{R}$ , we must have Y = 0. Hence,  $X \in \text{Lie}(K)$  if and only if  $X \in \mathfrak{k} = \ker \phi_*$ .

2.14. The General Linear Group. Let V be a vector space over C (what we say holds also over R). We write  $\mathfrak{gl}(V) = \mathfrak{gl}_{\mathbf{C}}(V) = \operatorname{End}_{\mathbf{C}}(V)$  for the vector space of C-linear endomorphisms of V. This is an associative algebra, hence it has a Lie algebra structure given by commutators: [X, Y] = XY - YX. Inside  $\mathfrak{gl}(V)$ , we have the locus  $\operatorname{GL}(V)$  consisting of invertible linear maps. This can also be described as  $\operatorname{GL}(V) = \{\det \neq 0\}$  where  $\det : \mathfrak{gl}(V) \to \mathbf{C}^*$  is the determinant. Since det is continuous, then  $\operatorname{GL}(V) \subset \mathfrak{gl}(V)$  is open, hence a manifold, hence a Lie group. Since  $\operatorname{GL}(V)$  is an open subset of a vector space, we may identify the tangent space  $T_{\mathrm{I}} \operatorname{GL}(V)$  with  $\mathfrak{gl}(V)$  itself.

**Proposition 2.28.** We have  $Lie(GL(V)) = \mathfrak{gl}(V)$ , in other words the Lie bracket of left invariant vector fields on GL(V) is the commutator bracket.

*Proof.* If  $X \in \mathfrak{gl}(V)$  and  $f \colon GL(V) \to \mathbf{R}$  is any function, then

$$\mathbf{X}f = \lim_{t \to 0} \frac{f(\mathbf{I} + t\mathbf{X}) - f(\mathbf{I})}{t}$$

Suppose f is actually the restriction of a *linear* function on  $\mathfrak{gl}(V)$ , then

$$\mathbf{X}f = \lim_{t \to 0} \frac{f(\mathbf{I} + t\mathbf{X}) - f(\mathbf{I})}{t} = f(\mathbf{X}).$$

Call  $\tilde{X}$ , the left invariant vector field corresponding to X. Recall this was defined by declaring  $\tilde{X}f(A) = X(f \circ L^A)$ , where  $A \in GL(V)$ . Explicitly,

$$\tilde{\mathbf{X}}f(\mathbf{A}) = \mathbf{X}(f \circ \mathbf{L}^{\mathbf{A}}) = \lim_{t \to 0} \frac{f \circ \mathbf{L}^{\mathbf{A}}(\mathbf{I} + t\mathbf{X}) - f \circ \mathbf{L}^{\mathbf{A}}(\mathbf{I})}{t} = \lim_{t \to 0} \frac{f(\mathbf{A} + t\mathbf{A}\mathbf{X}) - f(\mathbf{A})}{t}$$

Once again, if f is the restriction of a linear function on  $\mathfrak{gl}(V)$  then

$$\mathbf{X}f(\mathbf{A}) = f(\mathbf{A}\mathbf{X})$$

and since f is linear, Xf is still linear.

Suppose now X, Y  $\in \mathfrak{gl}(V)$ . Let us evaluate the Lie bracket  $[X, Y] \in \mathfrak{gl}(V)$  on linear functions. If f is linear and  $B \in GL(V)$ , we have

$$\begin{split} \tilde{\mathbf{X}}, \tilde{\mathbf{Y}}]f(\mathbf{B}) &= \tilde{\mathbf{X}}(\tilde{\mathbf{Y}}f)(\mathbf{B}) - \tilde{\mathbf{Y}}(\tilde{\mathbf{X}}f)(\mathbf{B}) \\ &= \tilde{\mathbf{Y}}f(\mathbf{B}\mathbf{X}) - \tilde{\mathbf{X}}f(\mathbf{B}\mathbf{Y}) \\ &= f(\mathbf{B}\mathbf{X}\mathbf{Y}) - f(\mathbf{B}\mathbf{Y}\mathbf{X}) = f(\mathbf{B}\mathbf{X}\mathbf{Y} - \mathbf{B}\mathbf{Y}\mathbf{X}). \end{split}$$

The Lie bracket is given by specializing B = I. To summarize, we've concluded that for all linear functions f we have [X, Y](f) = f(XY-YX). It follows that [X, Y] = XY-YX (if v, w are vectors, and for all elements f of the dual we have f(v) = f(w), then v = w).

We now prove that  $\exp$  is really  $\exp.$  If  $A\in\mathfrak{gl}(V),$  we define  $e^A$  to be the power series

$$e^{\mathbf{A}} = \sum_{k \ge 0} \frac{\mathbf{A}^k}{k!}$$

and one shows that it converges uniformly, defining an analytic function  $\mathfrak{gl}(V)\to\mathfrak{gl}(V).$  If  $B\in GL(V),$  then

$$\mathbf{B}e^{\mathbf{A}}\mathbf{B}^{-1} = e^{\mathbf{B}\mathbf{A}\mathbf{B}^{-1}}.$$

This is true as it is true for all partial sums  $\sum_{k=0}^{N}$  and it passes to the limit. We write  $Ad_B(A) = BAB^{-1}$ .

Let us specialize to the case of  $A \in \mathfrak{gl}(n, \mathbb{C})$ . We know there exists  $B \in GL(n, \mathbb{C})$ such that  $Ad_B A$  is upper triangular. If  $U \in \mathfrak{gl}(n, \mathbb{C})$  is upper triangular with diagonal entries  $a_1, \ldots, a_n$  then  $e^U$  is also upper triangular with diagonal entries  $e^{a_1}, \ldots, e^{a_n}$ . Hence  $e^U \in GL(n, \mathbb{C})$ .

**Proposition 2.29**. The exponential of a matrix is invertible:  $e : \mathfrak{gl}(V) \to GL(V)$ .

*Proof.* Pick a basis and work over C. Let  $A \in \mathfrak{gl}(n, \mathbb{C})$ . Find B such that  $\operatorname{Ad}_{B} A$  is upper triangular. Then  $e^{\operatorname{Ad}_{B} A} \in \operatorname{GL}(n, \mathbb{C})$ . But  $e^{\operatorname{Ad}_{B} A} = \operatorname{Ad}_{B} e^{A}$ , hence  $e^{A} \in \operatorname{GL}(n, \mathbb{C})$ .

Using the same trick, we see that

$$\det e^{\mathbf{A}} = e^{\operatorname{tr}\mathbf{A}}$$

(recall that for an upper triangular matrix U with diagonal entries  $a_i$  we have  $\det U = a_1 \cdots a_n$  and  $\operatorname{tr} U = a_1 + \cdots + a_n$ ).

**Proposition 2.30.**  $e^{A+B} = e^A e^B$  if AB = BA.

*Proof.* Call  $S_N(A) = \sum_{k=0}^N \frac{A^k}{k!}$  for the partial sum. We have  $e^A = \lim_{N\to\infty} S_N(A)$ . Since matrix multiplication is continuous, we have  $e^A e^B = \lim_{N\to\infty} S_N(A)S_N(B)$ . We want to show that  $\lim_{N\to\infty} S_N(A)S_N(B) - S_N(A + B) = 0$ . Notice  $S_0(A)S_0(B) - S_0(A + B) = I - I = 0$ ,  $S_1(A)S_1(B) - S_1(A + B) = (I + A)(I + B) - A - B = I + B + A + AB - (I + A + B) = AB$ .

$$\begin{split} S_2(A)S_2(B) - S_2(A+B) &= (I+A+\frac{A^2}{2})(I+B+\frac{B^2}{2}) - (I+A+B+\frac{(A+B)^2}{2}) \\ &= I+B+\frac{B^2}{2} + A + AB + \frac{AB^2}{2} + \frac{A^2}{2} + \frac{A^2B}{2} + \frac{A^2B^2}{4} \\ &- (I+A+B+\frac{A^2+AB+BA+B^2}{2}) \\ &= \frac{AB^2}{2} + \frac{A^2B}{2} + \frac{A^2B^2}{4} \end{split}$$

where we used that AB = BA. In general, one shows that We have

$$S_N(A)S_N(B) - S_N(A + B) = \sum \frac{B^l A^k}{l!k!}$$

where the sum is over all integers l, k such that  $1 \le l \le N$ ,  $1 \le k \le N$  and  $N + 1 \le l + k \le 2N$ . The point is that the denominators are factorials, which grow stupidly fast. Pick  $\mu > 1$  an upper bound for the entries of A and B. One shows that each entry of the RHS is bounded by

$$\sum \frac{n^{l+k-1}\mu^{l+k}}{l!k!} \le \frac{n\mu^{2N}N^2}{[N/2]!}$$

which goes to zero as  $N \to \infty$ .

**Theorem 2.31.** The exponential map  $\exp: \mathfrak{gl}(V) \to \operatorname{GL}(V)$  of the Lie group  $\operatorname{GL}(V)$  is given by  $A \mapsto e^A$ . In other words,  $\exp(A) = e^A$ .

**Proof.** Consider the map  $t \mapsto e^{tA}$ . This map is smooth and by differentiating the power series term by term we see that its derivative at t = 0 is A. By the previous Proposition we know that it is group homomorphism, i.e. a 1-parameter subgroup. By uniqueness,  $\exp_A(t) = e^{tA}$ . In particular,  $\exp(A) = \exp_A(1) = e^A$ .

2.15. Ad. Suppose G acts (smoothly) on the left on the manifold M. For  $g \in G$ , write  $\alpha_g \colon M \to M$  for the map which takes x to gx. In other words,  $\alpha_g$  sees the effect of the single element g on M. Now, if gx = x, then  $d\alpha_g \colon T_x M \to T_x M$  is an invertible linear map. Suppose now x is a fixed point. Then  $G \to GL(T_x M)$ , sending  $g \mapsto (\alpha_g)_* = d\alpha_g(e)$ , is a group homomorphism. By a *representation* of a group G we mean a Lie morphism  $G \to GL(V)$ .

A special case of this is the *adjoint representation*. The group G acts on itself by conjugation:  $C_g(x) = gxg^{-1}$ . Since  $e \in G$  is a fixed point, we may differentiate to obtain the adjoint representation Ad:  $G \to GL(\mathfrak{g})$ . By definition,  $Ad_g = (C_g)_* = d(C_g)(e)$ .

**Theorem 2.32.** Let G = GL(V). Then  $Ad_B A = BAB^{-1}$ .

*Proof.* Pick  $B \in GL(V)$ . If  $A \in \mathfrak{gl}(V)$ , we have

$$Ad_{B}(A) = (C_{B})_{*}A = d(C_{B})(I)A = \lim_{t \to 0} \frac{C_{B}(I + tA) - C_{B}(I)}{t} = BAB^{-1}. \qquad \Box$$

Suppose now  $\Pi \colon G \to GL(V)$  is a representation. Differentiating at the identity, we obtain a map of Lie algebras  $\pi = \Pi_* \colon \mathfrak{g} \to T_I \operatorname{GL}(V) = \operatorname{End}(V) = \mathfrak{gl}(V)$ . Just as group homomorphisms  $G \to GL(V)$  have a special name, Lie maps  $\mathfrak{g} \to \mathfrak{gl}(V)$  are called representations of Lie algebras.

**Proposition 2.33.** Let  $\Pi \colon G \to GL(V)$  and let  $\pi = \Pi_* \colon \mathfrak{g} \to \mathfrak{gl}(V)$ . Then

$$\pi(\mathbf{X}) = \frac{d}{dt} \Big|_{t=0} \Pi(\exp(t\mathbf{X}))$$

*Proof.* We want to compute  $\pi(X) = \Pi_* X$ . Consider the curve  $c(t) = \exp(tX)$ . We have c'(0) = X. Thus,  $\pi(X) = (\Pi \circ c)'(0)$ . Now,  $c \circ \Pi \colon \mathbf{R} \to \operatorname{GL}(V)$  hence its derivative as the explicit formula

$$\pi(\mathbf{X}) = (\Pi \circ c)'(0)$$

$$= \lim_{t \to 0} \frac{(\Pi \circ c)(t) - \mathbf{I}}{t}$$

$$= \lim_{t \to 0} \frac{\Pi(\exp t\mathbf{X}) - \mathbf{I}}{t}$$

$$= \frac{d}{dt}\Big|_{t=0} \Pi(\exp(t\mathbf{X})).$$

Once again, let's have a look at conjugation. Take the adjoint representation  $Ad: G \rightarrow GL(\mathfrak{g})$ . We may differentiate it, obtaining  $ad = (Ad)_*: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ . Unsurprisingly, this is called the *adjoint representation* of the Lie algebra  $\mathfrak{g}$ .

**Theorem 2.34.** Let  $X, Y \in \mathfrak{g} = Lie(G)$ . Then  $ad_X(Y) = [X, Y]$ .

*Proof.* We know that X is the velocity at t = 0 of the curve  $\exp tX$ , while Y is the velocity at s = 0 of the curve  $\exp sY$ . For  $g \in G$ ,  $\operatorname{Ad}_g(Y) = (C_g)_*Y$  is the velocity at s = 0 of the curve  $C_g(\exp sY)$ . While  $\operatorname{ad}_X = (\operatorname{Ad}_*)X$  is the velocity of t = 0 of the curve  $\operatorname{Ad}_{\exp tX}$ . Our goal is to show that the tangent vectors [X, Y],  $\operatorname{ad}_X Y \in \mathfrak{g}$  are the same. Hence we should show that they spit out the same value when we feed them a function  $f: G \to \mathbf{R}$ . Write

$$a(t, s) = C_{\exp tX}(\exp sY) = \exp(tX)\exp(sY)\exp(-tX)$$

so that

$$(\operatorname{ad}_{\mathbf{X}} \mathbf{Y})(f) = \frac{\partial^2}{\partial t \partial s} \Big|_{(0,0)} f \circ a(t,s)$$

By using the chain rule one shows that the RHS is equal to

$$(\mathrm{ad}_{\mathbf{X}}\,\mathbf{Y})(f) = \frac{\partial^2}{\partial t \partial s}\Big|_{(0,0)} f(\exp t\mathbf{X}\exp s\mathbf{Y}) - \frac{\partial^2}{\partial t \partial s}\Big|_{(0,0)} f(\exp s\mathbf{Y}\exp t\mathbf{X})$$

As usual, let  $\tilde{X},\tilde{Y}$  be the corresponding left invariant vector fields. If we show that

$$\frac{\partial^2}{\partial t \partial s}\Big|_{(0,0)} f(\exp t \mathbf{X} \exp s \mathbf{Y}) = \mathbf{X}(\tilde{\mathbf{Y}}f)$$

(and the same with X and Y swapped) we are done. But

$$\begin{split} (\tilde{\mathbf{Y}}f)(x) &= \mathbf{Y}(f \circ \mathbf{L}^x) \\ &= \frac{d}{ds}\Big|_{s=0} f \circ \mathbf{L}^x \circ \exp s\mathbf{Y} \\ &= \frac{d}{ds}\Big|_{s=0} (f(x \exp s\mathbf{Y})) \end{split}$$

So that

$$\mathcal{X}(\tilde{\mathcal{Y}}f) = \frac{d}{dt}\Big|_{t=0}\tilde{\mathcal{Y}}f(\exp t\mathcal{X}) = \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}(f(\exp t\mathcal{X}\exp s\mathcal{Y})).$$

Repeating the same with X and Y swapped we conclude.

By naturality of exp, we have a commutative diagram

$$\begin{array}{c} \mathfrak{g} \xrightarrow{\operatorname{ad}} \mathfrak{gl}(\mathfrak{g}) \\ \exp \downarrow \qquad \qquad \qquad \downarrow \exp \\ G \xrightarrow{\operatorname{Ad}} \operatorname{GL}(\mathfrak{g}) \end{array}$$

since  $ad = (Ad)_*$ . This means

$$Ad_{\exp X} Y = e^{ad_X} Y = Y + [X, Y] + \frac{[X, [X, Y]]}{2} + \frac{[X, [X, [X, Y]]]}{3!} + \cdots$$

Similarly, since conjugation  $\mathbf{C}_g$  is a group homomorphism, naturality of  $\exp$  applies and we have

$$\begin{array}{c} \mathfrak{g} \xrightarrow{\operatorname{Ad}_g} \mathfrak{g} \\ \stackrel{\operatorname{exp}}{\longrightarrow} \mathfrak{g} \\ \stackrel{\operatorname{C}_g}{\longrightarrow} \operatorname{G} \end{array} \end{array}$$

since, by definition,  $\mathrm{Ad}_g=(\mathrm{C}_g)_*.$  Elementwise, this means that for  $\mathrm{X}\in\mathfrak{g},$  we have

$$g \exp(t\mathbf{X})g^{-1} = \exp(\mathrm{Ad}_g t\mathbf{X})$$

**Theorem 2.35.** Let H < G, with both H and G connected Lie groups. Then H is normal if and only if  $\mathfrak{h} < \mathfrak{g}$  is an ideal.

An *ideal* of a Lie algebra  $\mathfrak{g}$  is a subspace such that  $[\mathfrak{g},\mathfrak{h}] \subset \mathfrak{h}$ .

*Proof.* Assume  $\mathfrak{h}$  is an ideal. Let  $X \in \mathfrak{g}$ ,  $Y \in \mathfrak{h}$  and let  $g = \exp X$ . Then

$$g(\exp \mathbf{Y})g^{-1} = \exp(\operatorname{Ad}_{g} \mathbf{Y})$$
  
=  $\exp((\exp \operatorname{ad}_{\mathbf{X}})(\mathbf{Y}))$   
=  $\exp\left(\mathbf{Y} + [\mathbf{X}, \mathbf{Y}] + \frac{[\mathbf{X}, [\mathbf{X}, \mathbf{Y}]]}{2!} + \cdots\right)$ 

Since  $\mathfrak{h}$  is an ideal, the series in the last term converges to an element in  $\mathfrak{h}$ , implying that  $g \exp Y g^{-1} \in H$ . But H is generated by elements of the form  $\exp Y$  and G is generated by elements of the form  $\exp X$ , which is enough to conclude.

Assume now H is normal. Let  $s, t \in \mathbf{R}$  and let  $Y \in \mathfrak{h}$ ,  $X \in \mathfrak{g}$ . We want to show that  $[X, Y] \in \mathfrak{h}$ . Let  $g = \exp tX$ . Once again

$$g(\exp s\mathbf{Y})g^{-1} = \exp(\mathrm{Ad}_g(s\mathbf{Y})) = \exp s\left((\exp \mathrm{ad}_{t\mathbf{X}})(\mathbf{Y})\right)$$

Let  $v = v(t) = (\exp \operatorname{ad}_{tX})(Y)$ . Since H is normal,  $g(\exp sY)g^{-1} \in H$ . Thus  $\exp sv \in H$  for all s, implying  $v \in \mathfrak{h}$ . But v(t) is a smooth curve in  $\mathfrak{h}$ , whose tangent vector at t = 0 is [X, Y]. Thus  $[X, Y] \in \mathfrak{h}$  as desired.

**Definition 2.36.** The *center* of G is the subgroup of elements commuting with everything. The *center* of  $\mathfrak{g}$  is  $Z(\mathfrak{g}) = \{X \in \mathfrak{g} \mid [X, Y] = 0 \forall Y \in \mathfrak{g}\}.$ 

**Theorem 2.37.** Let G be a connected Lie group. The center of G is the kernel of the adjoint representation.

*Proof.* Let g lie in the center of G and let  $X \in \mathfrak{g}$ . Then

(2.3) 
$$\exp t\mathbf{X} = g(\exp t\mathbf{X})g^{-1} = \exp t\operatorname{Ad}_{g}\mathbf{X}$$

for all  $t \in \mathbf{R}$ . Thus  $X = Ad_g X$  for all X, or  $Ad_g = I$ . In other words  $g \in ker(Ad)$ .

Conversely, let  $g \in \ker \operatorname{Ad}$ . Then 2.3 holds, when read backwards. Hence g commutes with all elements of the form  $\exp tX$ , and since G is connected we conclude that  $g \in Z(G)$ .

**Remark 2.38.** In particular, if G is a Lie group with trivial center, then  $Ad: G \to GL(\mathfrak{g})$  realizes G as a matrix group.

**Corollary 2.39.** Let G be a connected Lie group. Then its center Z(G) is a closed Lie subgroup, with  $Lie(Z(G)) = Z(\mathfrak{g})$ .

*Proof.* The previous theorem tells us that Z(G) consists of ker Ad. But we know that for a Lie map  $G \to H$  its kernel is a closed Lie subgroup with Lie algebras the kernel of the derivative. And  $d \operatorname{Ad} = \operatorname{ad}$ .

A Lie algebra is called *abelian* if it has trivial bracket.

Corollary 2.40. A connected Lie group is abelian if and only if its Lie algebra is abelian.

**Proposition 2.41.** Let  $X, Y \in \mathfrak{g}$  for a Lie group G. If [X, Y] = 0 then  $\exp X \exp Y = \exp(X + Y) = \exp Y \exp X$ .

*Proof.* Consider the subspace  $\mathfrak{h} < \mathfrak{g}$  generated by X, Y. Since they commute, it's an abelian subalgebra. To it corresponds an abelian connected Lie subgroup H < G. Consider  $\alpha(t) = \exp(tX) \exp(tY)$ , which is a 1-parameter subgroup since it is smooth and H is abelian. The tangent vector at t = 0 is given by X + Y. By uniqueness, we have  $\exp(tX) \exp(tY) = \exp t(X + Y)$ .

In the proof, we used the following.

**Lemma 2.42.** Let G be a Lie group and let  $\alpha(t), \beta(t)$  be smooth curves in G, such that  $\alpha(0) = e = \beta(0)$ . Let  $\gamma(t) = \alpha(t)\beta(t)$  be the curve obtained via multiplication. Then  $\gamma'(0) = \alpha'(0) + \beta'(0)$ .

**Proof.** Let  $m: G \times G \to G$  denote multiplication. By the chain rule, it suffices to show that  $m_*(v, w) = v + w$ , for  $v, w \in T_eG$  and  $m_* = dm(e, e)$  the derivative at  $(e, e) \in G \times G$ . Since (v, w) = (v, 0) + (0, w), we reduce to computing  $m_*(v, 0)$ . Explicitly,  $v \in T_eG$  and  $(v, 0) = i_*v$  where  $i: G \to G \times G$  is the inclusion  $g \mapsto (g, e)$ . But  $m \circ i$  is the identity, hence we are done.

Here is a common way to produce Lie groups.

**Proposition 2.43.** Let  $\beta$  be a bilinear form on a vector space V (over **R** or **C**). Let G be

$$\operatorname{Aut}(\mathbf{V},\beta) = \{\mathbf{A} \in \operatorname{GL}(\mathbf{V}) \mid \beta(\mathbf{A}v,\mathbf{A}w) = \beta(v,w)\} < \operatorname{GL}(\mathbf{V}).$$

Let  $\mathfrak{g}$  be the space of derivations

$$Der(\mathbf{V},\beta) = \{\mathbf{X} \in \mathfrak{gl}(\mathbf{V}) \mid \beta(\mathbf{X}v,w) + \beta(v,\mathbf{X}w) = 0\} < \mathfrak{gl}(\mathbf{V}).$$

Then G is a Lie group with  $Lie(G) = \mathfrak{g}$ .

*Proof.* First off, G is a closed subgroup of GL(V), hence a Lie group. Secondly,  $\mathfrak{g}$  is a Lie subalgebra of  $\mathfrak{gl}(V)$ . Let now  $X \in \mathfrak{gl}(V)$ .

If  $X \in \mathfrak{g}$ , then  $\exp tX \in G$  for all  $t \in \mathbb{R}$ . Indeed, if  $A = e^{tX}$ , then a bunch of cancellations happen in  $\beta(Av, Aw)$  making it equal to  $\beta(v, w)$ . This implies  $X \in \operatorname{Lie}(G)$ , hence  $\mathfrak{g} \subset \operatorname{Lie}(G)$ .

Conversely, suppose  $X \in Lie(G)$ . In other words,  $\exp tX \in G$  for all  $t \in \mathbf{R}$ . Explicitly, if  $A_t = \exp tX$ ,  $\beta(A_tv, A_tw) = \beta(v, w)$  for all  $v, w \in V$ . Then

$$0 = \frac{a}{dt}\Big|_{t=0} \beta(\mathbf{A}_t v, \mathbf{A}_t w)$$
  
=  $\beta(\frac{d}{dt}\Big|_{t=0} \mathbf{A}_t v, w) + \beta(v, \frac{d}{dt}\Big|_{t=0} \mathbf{A}_t w)$   
=  $\beta(\mathbf{X}v, w) + \beta(v, \mathbf{X}w)$ 

and we are done.

In the proof we used the following lemma.

**Lemma 2.44.** Let  $\gamma(t), \delta(t)$  be two curves in a vector space V. Let  $\beta$  be a bilinear form on V. Then

$$\frac{d}{dt}\Big|_{t=0}\beta(\gamma(t),\delta(t)) = \beta(\gamma'(0),\delta(0)) + \beta(\gamma(0),\delta'(0)).$$

Proof. By definition

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\beta(\gamma(t),\delta(t)) &= \lim_{t\to0} \frac{\beta(\gamma(t),\delta(t)) - \beta(\gamma(0),\delta(0))}{t} \\ &= \lim_{t\to0} \frac{\beta(\gamma(t),\delta(t)) - \beta(\gamma(0),\delta(t)) + \beta(\gamma(0),\delta(t)) - \beta(\gamma(0),\delta(0))}{t} \\ &= \lim_{t\to0} \beta(\frac{\gamma(t) - \gamma(0)}{t},\delta(t)) + \beta(\gamma(0),\delta'(0)) \\ &= \beta(\gamma'(0),\delta(0)) + \beta(\gamma(0),\delta'(0)). \end{aligned}$$

A slight variation of this is if we have a bilinear map  $b: V \otimes V \to V$ . Then the Lie group of matrices preserving b has for Lie algebra the space of derivations:  $A \in \mathfrak{gl}(V)$  such that Ab(v, w) = b(Av, w) + b(v, Aw). This can be applied to the following setup.

**Remark 2.45.** Let  $\mathfrak{g}$  be a Lie algebra. The bracket is a bilinear map  $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ . The group of linear isomorphisms preserving the bracket are precisely the Lie algebra automorphisms of  $\mathfrak{g}$ ,  $\operatorname{Aut}(\mathfrak{g})$ . From the previous paragraph it follows that  $\operatorname{Aut}(\mathfrak{g})$  is a Lie group, with Lie algebra given by the derivations with respect to the bracket.

Let G be a simply connected Lie group. Then Aut(G), the group of invertible Lie morphisms, is in bijection with  $Aut(\mathfrak{g})$ . From the discussion above, Aut(G) becomes a Lie group, with Lie algebra the derivations of  $\mathfrak{g}$ .

2.16. **Examples.** The most common Lie groups are *matrix groups*, i.e. closed subgroups of GL(V) for V a vector space. We have seen already that for b a bilinear form, the group Aut(V, b) < GL(V) is a Lie group with Lie algebra given by derivations.

Here are a few instances of this. The orthogonal group and the complex orthogonal group

$$O(n) = \{ \mathbf{A} \in \operatorname{GL}(n, \mathbf{R}) \mid \mathbf{A}^{t} \mathbf{A} = \mathbf{I} \}$$
$$O(n, \mathbf{C}) = \{ \mathbf{A} \in \operatorname{GL}(n, \mathbf{C}) \mid \mathbf{A}^{t} \mathbf{A} = \mathbf{I} \}$$

whose Lie algebras

$$\mathbf{o}(n) = \{ \mathbf{A} \in \mathfrak{gl}(n, \mathbf{R}) \mid \mathbf{A}^t + \mathbf{A} = 0 \}$$
$$\mathbf{o}(n, \mathbf{C}) = \{ \mathbf{A} \in \mathfrak{gl}(n, \mathbf{C}) \mid \mathbf{A}^t + \mathbf{A} = 0 \}.$$

consist of skew-symmetric matrices. The unitary group

$$U(n) = \{ \mathbf{A} \in \operatorname{GL}(n, \mathbf{C}) \mid \bar{\mathbf{A}}^t \mathbf{A} = \mathbf{I} \}$$
$$\mathfrak{u}(n) = \{ \mathbf{A} \in \mathfrak{gl}(n, \mathbf{C}) \mid \bar{\mathbf{A}}^t + \mathbf{A} = 0 \}$$

whose Lie algebra consists of skew-hermitian matrices.

The map det:  $GL(V) \rightarrow C^*$  is a Lie morphism. Its kernel is SL(V), the *special linear* group. Since it's a closed subgroup it's a Lie group. Its Lie algebra is given by the kernel of det<sub>\*</sub>, the derivative at the identity of det. One can compute that det<sub>\*</sub> = tr, the trace. Hence the Lie algebra

$$\mathfrak{sl}(\mathbf{V}) = \{\mathbf{A} \in \mathfrak{gl}(\mathbf{V}) \mid tr(\mathbf{A}) = 0\} < \mathfrak{gl}(\mathbf{V})$$

consists of traceless matrices. Another way to see this is to use the exponential. Indeed, we know that

$$\det e^{\mathbf{A}} = e^{\operatorname{tr}\mathbf{A}}.$$

If tr A = 0 we have det  $e^{A} = 1$ . On the other hand, if det  $e^{tA} = 1$  it implies that  $tr(tA) \in 2\pi i \mathbb{Z}$ . Since at t = 0 it is zero, it follows it must be constantly zero.

In addition to inner products, we also like symplectic forms. On  $\mathbb{R}^{2n}$  there is the standard symplectic form

$$\omega(x,y) = \sum_{j=1}^n x_j y_{n+j} - x_{n+j} y_j.$$

The corresponding Lie group is the *real symplectic group*  $Sp(n, \mathbf{R})$ . If

$$\Omega = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}$$

with I being the  $n \times n$  identity matrix, then

$$\operatorname{Sp}(n, \mathbf{R}) = \{ \mathbf{A} \in \operatorname{GL}(2n, \mathbf{R}) \mid \mathbf{A}^t \Omega \mathbf{A} = \Omega \}$$

and its Lie algebra is given by

$$\mathfrak{sp}(n, \mathbf{R}) = \{ \mathbf{A} \in \mathrm{GL}(2n, \mathbf{R}) \mid \Omega \mathbf{A} + \mathbf{A}^t \Omega = 0 \}.$$

Similarly, the *complex symplectic group* and algebra

$$Sp(n, \mathbf{C}) = \{ \mathbf{A} \in GL(2n, \mathbf{C}) \mid \mathbf{A}^{t} \Omega \mathbf{A} = \Omega \}$$
  
$$\mathfrak{sp}(n, \mathbf{C}) = \{ \mathbf{A} \in GL(2n, \mathbf{C}) \mid \Omega \mathbf{A} + \mathbf{A}^{t} \Omega = 0 \}.$$

Finally, we have the compact symplectic group  $Sp(n) = Sp(n, \mathbb{C}) \cap U(2n)$ .

Combining SL with the previous groups leads to more interesting examples. The *special* orthogonal and unitary groups

$$SO(n) = O(n) \cap SL(n, \mathbf{R})$$
$$SU(n) = U(n) \cap SL(n, \mathbf{C})$$

and, while  $\mathfrak{so}(n) = \mathfrak{o}(n)$ 

$$\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n) = \{ \mathbf{A} \in \mathfrak{gl}(n, \mathbf{C}) \mid \bar{\mathbf{A}}^t + \mathbf{A} = 0 = \operatorname{tr}(\mathbf{A}) \}.$$

This will make more sense later, when we investigate these groups further.

2.17. Homogeneous spaces. A space M is called *homogeneous* if there is a Lie group G acting transitively on it. If  $m \in M$  and H is the stabilizer group, then M is diffeomorphic to G/H, in a G-equivariant way.

**Theorem 2.46.** Let H be a closed subgroup of a Lie group G. Let  $\pi: G \to G/H$  be the quotient. Then G/H admits a manifold structure which may be characterized as the unique one such that  $\pi$  is smooth and has local sections. When H is normal, G/H is also Lie group.

No proof.

**Theorem 2.47.** Let  $G \times M \to M$  be a (smooth) transitive action of a Lie group on a manifold. Pick a point  $m \in M$  and let H be its stabilizer (aka isotropy) group. Then  $G/H \to M$  is a G-equivariant diffeomorphism.

Again, no proof.

2.17.1. Spheres. Let's give some examples. Of course we have the action  $GL(n, \mathbf{R}) \times \mathbf{R}^n \to \mathbf{R}^n$ , which restricts to  $O(n) \times S^{n-1} \to S^{n-1}$ . This action is transitive. One checks that the stabilizer group of the last canonical vector  $e_n$  is O(n-1), where a matrix  $A \in O(n-1)$  is sent to

$$\begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Thus,

$$\mathbf{S}^{n-1} = \mathbf{O}(n) / \mathbf{O}(n-1).$$

The same discussion may be repeated with SO, concluding that

$$\mathbf{S}^{n-1} = \mathbf{SO}(n) / \mathbf{SO}(n-1).$$

Switching to the unitary groups, an entirely parallel discussion over  $\mathbb{C}^n$  shows that  $S^{2n-1} = U(n)/U(n-1) = SU(n)/SU(n-1)$ . Since  $SU(1) = \{e\}$ , we see that  $SU(2) = S^3$ .

**Theorem 2.48.** The only spheres that are Lie groups are  $S^1$  and  $S^3$ .

We give a very rough sketch, stolen from mathoverflow.

*Proof.* Let G be both a sphere and a Lie group. Let  $n = \dim G$  and  $\mathfrak{g} = \text{Lie}(G)$ . Suppose G is abelian, then  $\mathfrak{g}$  is abelian, meaning  $\mathfrak{g} \simeq \text{Lie}(\mathbb{R}^n)$ . The isomorphism  $\text{Lie}(\mathbb{R}^n) \to \mathfrak{g}$  extends to a Lie map  $\mathbb{R}^n \to G$ , which must be a covering space. In other words, G has contractible universal cover. Hence  $G \simeq S^1$ .

Let G be non abelian. Since G is compact, there is an Ad-invariant inner product  $\langle,\rangle$  on g. Consider the map

$$\tau \colon \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}(x, y, z) \mapsto \tau(x, y, z) = \langle [x, y], z \rangle$$

which can be shown to be a 3-form on g. Since G is non-abelian, there exist x, y such that  $[x, y] \neq 0$ . Thus,  $\tau(x, y, [x, y]) = |[x, y]|^2 \neq 0$ . Extend  $\tau$  to a non-zero 3-form by left translation. One then shows that this form is also right invariant. One can show that bi-invariant forms are actually closed. Suppose  $\tau$  is a boundary, i.e.  $\tau = d\sigma$ . Then one can show that  $\sigma$  can be chosen to be itself bi-invariant. But bi-invariant forms are closed, hence  $\tau = 0$  which is a contradiction. Hence  $H^3(G, \mathbf{R}) \neq 0$ . Thus,  $G \simeq S^3$ .  $\Box$ 

2.17.2. Projective space. We write  $\mathbf{RP}^n = (\mathbf{R}^{n+1} \setminus \{0\})/\mathbf{R}^* = S^n/(\mathbf{Z}/2\mathbf{Z})$  for the *n*-dimensional real projective space. Running yet again the same argument, one sees that

$$\mathbf{RP}^{n-1} = \mathrm{SO}(n) / \mathrm{O}(n-1)$$

where this time a matrix  $A \in O(n-1)$  is sent to

$$\begin{pmatrix} & & 0 \\ & A & \vdots \\ & & 0 \\ 0 & \cdots & 0 & \frac{1}{\det A} \end{pmatrix}$$

in SO(n).

Finally,

$$\mathbf{CP}^{n-1} = \mathrm{SU}(n) / \mathrm{U}(n-1)$$

where a matrix  $A \in U(n-1)$  is sent to

$$\begin{pmatrix} & & 0 \\ A & \vdots \\ & & 0 \\ 0 & \cdots & 0 & \frac{1}{\det A} \end{pmatrix}$$

2.17.3. Stiefel manifolds. Let V be a vector space and let  $p \leq \dim V = n$ . The Stiefel manifold  $S_p(V)$  consists of ordered tuples  $(v_1, \ldots, v_p)$  of linearly independent vectors in V. Now, GL(V) clearly acts on  $S_p(V)$  and the action is obviously transitive. If we fix a point  $(v_1, \ldots, v_p) \in S_p(V)$  then its stabilizer consists of those linear maps which keep the  $v_i$  fixed. By choosing coordinates, we may assume  $v_i = e_i$  so that the stabilizer becomes the subgroup

$$\mathbf{H} = \left\{ \begin{pmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \in \mathrm{GL}(n, \mathbf{R}) \right\}$$

where I is the  $p \times p$  identity matrix. Thus  $S_p(\mathbf{R}^n)$  is the homogeneous space  $GL(n, \mathbf{R})/H$ . One checks that the manifold structure on  $S_p(V)$  does not depend on any choices.

2.17.4. Grassmannians. Let Gr(k, n) be the set of k-dimensional subspaces of  $\mathbb{R}^n$ . Obviously  $GL(n, \mathbb{R})$  acts on it. But also O(n) acts on it. Consider the subspace spanned by  $e_1, \dots, e_k$ . The stabilizer of the O(n) action is given by the subgroup

$$\mathbf{H} = \left\{ \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} \in \mathbf{O}(n) \mid \mathbf{A} \in \mathbf{O}(k), \mathbf{B} \in \mathbf{O}(n-k) \right\}$$

We may identify H with  $O(k) \times O(n-k)$ . Hence

$$\operatorname{Gr}(k, n) = \operatorname{O}(n) / (\operatorname{O}(k) \times \operatorname{O}(n-k))$$

One should also check that the choices we made don't affect anything.

#### 2.18. More about the examples.

**Proposition 2.49.** Let H be a closed subgroup of a Lie group G. If H and G/H are connected, then G is connected.

*Proof.* This follows from the long exact sequence of a fibration, but it can also be proved by hand.  $\Box$ 

**Theorem 2.50.** The groups SO(n), SU(n), U(n) are connected.

*Proof.* SO(1) and SU(1) consist of one point, hence they are connected.  $U(1) = S^1$  is also connected. Since spheres are connected, we may use the previous proposition.  $\Box$ 

**Theorem 2.51.** The Lie group O(n) has two connected components.

*Proof.* Let  $A \in O(n)$ , by definition  $AA^t = I$ , hence  $(\det A)^2 = 1$  which means  $\det A = \pm 1$ . Let  $\Sigma$  be diag $(-1, 1, \ldots, 1)$ . If det A = -1, then  $\Sigma A \in SO(n)$ . Hence  $O(n) = SO(n) \cup \Sigma SO(n)$ .

**Theorem 2.52.** The group  $GL(n, \mathbf{R})$  has two connected components. In particular,  $GL(n, \mathbf{R}) = GL^+(n, \mathbf{R}) \times \mathbf{Z}/2\mathbf{Z}$ .

*Proof.* If  $A \in GL(n, \mathbb{R})$  then det A is either positive or negative. If  $\Sigma = \operatorname{diag}(-1, 1, \ldots, 1)$  as before, then det  $\Sigma A = -\det A$ . Hence, the set of matrices with negative determinant is diffeomorphic to the subgroup  $GL^+(n, \mathbb{R})$  of matrices with positive determinant. It suffices to show that  $GL^+(n, \mathbb{R})$  is connected. For each A, we show there is a continuous path from A to the identity I, showing  $GL^+(n, \mathbb{R})$  is path-connected, hence connected. Let  $A \in GL^+(n, \mathbb{R})$  and let A = PR be a polar decomposition. We must have det R = 1. Notice that  $P_t = tI + (1-t)P$  is symmetric positive-definite for each t, so  $P_tR$  is a path connecting A to R. But SO(n) is connected, hence we are done.

**Lemma 2.53.** Polar decomposition: if  $A \in GL(n, \mathbf{R})$  then A = PR, with P positive definite symmetric and  $R \in O(n)$ .

*Proof.* All the eigenvalues of a symmetric matrix are real and *positive-definite* means all the eigenvalues are positive. Take  $A \in GL(n, \mathbf{R})$ . The matrix  $B = (AA^t)^t = AA^t$  is symmetric. Let a be an eigenvalue for B with eigenvector v. Then

$$\begin{split} a \langle v, v \rangle &= \langle \mathbf{B} v, v \rangle \\ &= \langle \mathbf{A} \mathbf{A}^t v, v \rangle \\ &= \langle \mathbf{A}^t v, \mathbf{A}^t v \rangle \geq 0 \end{split}$$

hence  $a \ge 0$ . Since det  $A \ne 0$ , a > 0. Thus B is a symmetric positive-definite matrix. Since it is symmetric, there exists  $C \in O(n)$  which diagonalizes it (this is the spectral theorem). Hence,  $CBC^t$  is a diagonal matrix with positive eigenvalues: we may take its (positive) square root, call it D. We record for the later the fact that

$$\mathbf{B} = \mathbf{C}^t \mathbf{D}^2 \mathbf{C} = \mathbf{C}^t \mathbf{D} \mathbf{C} \mathbf{C}^t \mathbf{D} \mathbf{C} = (\mathbf{C}^t \mathbf{D} \mathbf{C})^2.$$

Let  $P = C^t DC$ . Let  $R = P^{-1}A$ . Certainly P is a positive-definite symmetric matrix. To show  $R \in O(n)$  we compute

$$\begin{aligned} \mathbf{R}\mathbf{R}^{t} &= \mathbf{P}^{-1}\mathbf{A}\mathbf{A}^{t}(\mathbf{P}^{-1})^{t} \\ &= \mathbf{P}^{-1}\mathbf{B}(\mathbf{P}^{t})^{-1} \\ &= \mathbf{P}^{-1}\mathbf{B}\mathbf{P}^{-1} \\ &= \mathbf{P}^{-1}(\mathbf{C}^{t}\mathbf{D}\mathbf{C})^{2}\mathbf{P}^{-1} \\ &= \mathbf{P}^{-1}\mathbf{P}^{2}\mathbf{P}^{-1} = \mathbf{I}. \end{aligned}$$

2.19. Loose ends. We conclude with a few facts found in the exercises of [War83].

- Exp for  $GL(n, \mathbf{R})$  is not surjective (example on page 134).
- A description of the only 2D non abelian Lie algebra, the only 2D simply connected Lie group.
- An example of matrices whose exps don't commute.
- Exp for  $GL(n, \mathbf{C})$  is surjective.
- Abelian Lie groups are all  $\mathbf{R}^n \times \mathbf{T}^m$ .
- $U(n) = S^1 \times SU(n)$
- $GL(n, \mathbf{C})$  is connected.

## 3. Outroduction

All right, that was fun. Now let's move on to [Hal15].

### References

- [Hall5] Brian Hall, Lie groups, Lie algebras, and representations, second ed., Graduate Texts in Mathematics, vol. 222, Springer, Cham, 2015, An elementary introduction.
- [War83] Frank W. Warner, Foundations of differentiable manifolds and Lie groups, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983, Corrected reprint of the 1971 edition.