This is a nice theorem to state:

"A variety is completely recovered by its category of coherent sheaves."

Let's be more precise:

Theorem (Gabriel) Let X,Y be two varieties, then X is isomorphic to Y if and only if the category Coh(X) is equivalent to Coh(Y).

This theorem has seen many extensions. The most general* form works for any quasi-separated scheme [Gabriel, Rosenberg, Gabber/Brandenburg].

*If one is willing to think of Coh(X) together with its tensor product, i.e. if one considers the *monoidal* category (Coh(X), ⊗), then one has a corresponding "Tannakian-flavored" variant of the reconstruction theorem [Ballman, Lurie, Brandenburg]. This version needs much more information from Coh(X) but has the big advantage of working even for stacks (unlike the ordinary one).

The idea is to extract a scheme $Z(\mathcal{A})$ out of an abelian category $\mathcal{A}$, such that $Z(\mathrm{Coh}(X)) = X$ (by default, if $\mathcal{A}=\mathcal{B}$ then $Z(\mathcal{A})=Z(\mathcal{B})$).

The original approach was to start by first producing a topological space $|\mathcal{A}|$, so that $|\mathrm{Coh}(X)| = |X|$ - the topological space underlying X.

The key fact is the bijection:

$$\{\text{closed irreducible subsets of } |X|\} \longleftrightarrow \{\text{Serre subcategories of } \mathrm{Coh}(X)\}$$

$$Y \subset |X| \longmapsto \mathrm{Coh}_Y(X)$$

where *Serre* means "closed under subobjects, quotients and extensions" and $\mathrm{Coh}_Y(X)$ is the category of sheaves supported on Y.

The point is that Serre subcategories make sense in *any* abelian category.

With extra work, one produces the structure sheaf $\mathcal{O}_X$, thus recovering the whole scheme X.

But there should be another way to prove this theorem:

$$X = \mathrm{Hilb}^1(X) = \text{moduli space of point-looking sheaves.}$$

Nowadays we are all into moduli, so it would be nice if we could write:

"X is the moduli space of *points* of Coh(X)."

Indeed, this is possible. Here are some of the advantages:

- a new paper on the arXiv,
- works for algebraic spaces and not just schemes,
- works for twisted varieties - where the twist comes from *any* class in $H^2(X, \mathcal{O}_X)$.

Main disadvantage: works at most for quasi-compact and separated spaces (it is very likely that this approach simply does not apply to the non-separated world).

So, what is a pointlike object of an abelian category? How do we build a moduli space of them?

Let's take a step back and review something we know well. There are at least two ways to present a scheme:

- X = ringed space = topological space + sheaf of rings = $(|X|, \mathcal{O}_X)$
- X = moduli space = collection of maps $S \to X$, with S affine scheme = the functor $\mathrm{Hom}(-,X)$.

As we like moduli spaces we prefer the second approach, by default.*

*Hilbert schemes illustrate this principle quite well. Hilb(X) is naturally a functor and for good reason. Hironaka gave an example of a non-projective threefold whose Hilbert scheme of points is not a scheme, but rather an algebraic space. Algebraic spaces sit in between schemes and stacks and they do *not* admit a description in terms of ringed spaces, so we are stuck with functors (or maybe topoi, but no one wants to work with those).

The idea is to extract a moduli space $\mathrm{Pt}_{\mathcal{A}}$ from an abelian category $\mathcal{A}$, such that $\mathrm{Pt}_{\mathrm{Coh}(X)}=X$.
How? One needs to define what a family of pointlike objects is.

Let's start with $\mathcal{A}=\mathrm{Coh}(X)$ and do some reverse engineering.

As $\mathrm{Pt}_{\mathrm{Coh}(X)}=X$, an S-family of pointlike objects in $\mathrm{Coh}(X)$ is nothing but a morphism $S\to X$.

A morphism is equivalent to its graph $\Gamma\subset S\times X$, which is a closed* subscheme of $S\times X$.

In turn, this is equivalent to the structure sheaf $\mathcal{O}_\Gamma$.

A "family of points" is the structure sheaf a graph.

*Here is where separatedness of X separated creeps in .

So, $\mathrm{Pt}_{\mathrm{Coh}(X)}(S) = \{\mathcal{O}_\Gamma$, for Γ the graph of a morphism $S \longrightarrow X\}$.

Can "being a graph" be phrased categorically? Yes - diagram please.

$$S \longleftarrow S \times X \longrightarrow X$$

Given a morphism $S \to X$, the key property is the bijection

$$\{\text{subschemes of } S\} \longleftrightarrow \{\text{quotients of } \mathcal{O}_\Gamma\}$$

given by $\mathrm{pr}_S{}^*(-) \otimes \mathcal{O}_\Gamma$. Why?

Let's denote by $\mathrm{gr}: S \to S \times X$ the graph morphism corresponding to $S \to X$.
The bijection follows once we notice that

$$\mathrm{pr}_S \circ \mathrm{gr} = \mathrm{id}_S, \quad \mathcal{O}_\Gamma = \mathrm{gr}_* \mathcal{O}_S, \quad \mathrm{gr}_*(-) = \mathrm{pr}_S{}^*(-) \otimes \mathcal{O}_\Gamma.$$

Now is time for the general definition. A quasi-coherent sheaf F on $S \times X$ is a graph* if and only if:

1. $pr_S^*(-) \otimes F$ induces a bijection between subschemes of S and quotients of F;
2. F is flat over S and of finite type;
3. for all $G \in Coh(X)$ we have $Hom(pr_X^*(G),F) \in Coh(S)$;
4. $M \longrightarrow Hom(F, F \otimes pr_S^* M)$ is an isomorphism, for all $M \in Coh(S)$.

All these properties make sense in *any* abelian category. So, $Pt_{\mathcal{A}}$ is well defined and $Pt_{Coh(X)} = X$. Hence, we have reproved Gabriel's theorem.

*This definition is technical, but the main bit is property 1. To deal with non-noetherian spaces one needs to slightly modify 3.

Actually, I lied: an F satisfying 1-4 is only a graph up to a twist of a line bundle $L \in Pic(S)$.

$Pt_{Coh(X)}$ is not X, but rather the trivial G_m-gerbe on X.

Any $\alpha \in H^2(X, G_m)$, defines a category $Coh(X,\alpha)$ of α-twisted sheaves.

It turns out that $Pt_{Coh(X,\alpha)} = \alpha$, i.e. the moduli of points of α-twisted sheaves is the gerbe corresponding to the twist α.

Theorem* Let X,Y be two varieties and let α,β be two classes in H^2. Then $Coh(X,\alpha)$ is equivalent to $Coh(Y,\beta)$ if and only if there exists an isomorphism $g: X \to Y$, such that $g^*\beta = \alpha$.

*This twisted reconstruction theorem was already studied by Perego, Canonaco-Stellari and Antieau.

Of course a question remains: if $\mathcal{A}$ is some other abelian category, what is $\mathrm{Pt}_{\mathcal{A}}$?

Anyway, that's all. This wasn't really a poster, but rather a bunch of slides next to each other (at least it had lots of colors).

If you are interested, the relevant paper is:
Moduli Problems in Abelian Categories and the Reconstruction Theorem
John Calabrese (Rice)
Michael Groechenig (Imperial College).