Groups also have representations

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Abstract. In the Fall of 2017 I taught out of Hall's excellent book on Lie theory. Part I was covered with great detail (with the exception of the BCH formula), Part II was more rushed, and we had zero time left for Part III. In the last three lectures I decided to summarize the main results of this last part. So here is a summary of that summary.

All mistakes appearing here are mine and mine only: you cannot have them.

1. Finite groups

Although not part of the book, let us briefly summarize some elements of the representation theory of finite groups. So, let G be a finite group. We want to study finite-dimensional complex representations.

Theorem 1.1 (Maschke). Representations are completely reducible.

Proposition 1.2 (Schur's lemma). Let V, W be two irreps. Then $\mathrm{Hom}_{\mathrm{G}}(\mathrm{V}, \mathrm{W})$ is either zero or isomorphic to $\mathbf{C}$. In particular, any non-zero intertwiner $\mathrm{V} \to \mathrm{W}$ is an isomorphism (of representations).

At the very least, what we would like next is a way to construct all irreps. For this purpose, it is useful to introduce the group algebra.

1.1. **Group algebra**.
Let $\mathbf{C}[\mathrm{G}]$ be the set of functions $\phi\colon \mathrm{G} \to \mathbf{C}$. This set becomes an associative (unital) $\mathbf{C}$-algebra by defining the *convolution*

$$(\phi * \psi)(x) = \sum_{y \in \mathrm{G}} \phi(x)\psi(y^{-1}x).$$

If G is not abelian, then $\mathbf{C}[\mathrm{G}]$ is not commutative. As a vector space, $\mathbf{C}[\mathrm{G}] \simeq \mathbf{C}^{|\mathrm{G}|}$.

Proposition 1.3. $\mathtt{Mod}(\mathbf{C}[\mathrm{G}])$ is the same thing as $\mathtt{Rep}(\mathrm{G})$.

Also, *irreducible* representations correspond to what are called *simple* modules, i.e. modules with no non-trivial submodules. Suppose M is such a module, i.e. a G-irrep. Let $0 \neq v \in \mathrm{M}$. There is a map $\mathbf{C}[\mathrm{G}] \to \mathrm{M}$ sending $1 \mapsto v$. Since M is simple, this map is surjective. So all irreducible representations may be obtained as quotients of $\mathbf{C}[\mathrm{G}]$.

Now, $\mathbf{C}[\mathrm{G}]$ itself may be viewed as a G-rep, called the *regular representation.* Maschke's therem tells us that $\mathbf{C}[\mathrm{G}] = \oplus_i \mathrm{W}_i$, where i runs over a finite set and W_i is an irreducible representation. So, let M be a random irrep. We have:

$$0 \neq 0\,\mathrm{Hom}_{\mathrm{G}}(\mathbf{C}[\mathrm{G}], \mathrm{M}) = \oplus_i \mathrm{Hom}_{\mathrm{G}}(\mathrm{W}_i, \mathrm{M}).$$

where the LHS is nonzero as we can always find a surjection $\mathbf{C}[\mathrm{G}] \twoheadrightarrow \mathrm{M}$. Schur's lemma now tells us there must be at least one i, such that $\mathrm{W}_i \simeq \mathrm{M}$. This means that:

> *All* irreps show up as factors of the regular representation. As a consequence, there are at most $|\mathrm{G}|$ distinct G-irreps.

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1.2. **Characters.** It would be nice to actually know the number of irreps. Turns out that characters are the way to go. If (Π, V) is a finite-dimensional (complex) representation, its *character* is the function

$$\chi_\Pi \colon G \to \mathbf{C}$$
$$\chi_\Pi(g) = \mathrm{tr}(\Pi(g))$$

Remark 1.4. There is a mild clash in terminology. Sometimes, by a character of a group it is meant a group homomorphism $G \to \mathbf{C}^*$. [Since G is finite, every element has finite order and hence must be a root of unity. So characters in this latter sense are the same as group homomomrphisms $G \to S^1 = U(1)$.] Given a group hom $X \colon G \to \mathbf{C}^*$, we obtain a one-dimensional representation $G \to \mathrm{End}(\mathbf{C})$ by declaring $\Pi(g)(z) = X(g)z$. Obviously, $\chi_\Pi(g) = X(g)$ for all g. So characters in this latter sense are the same as characters of one-dimensional representations.

Proposition 1.5. Let $(\Pi, V), (\Sigma, W)$ be two representations. Then

- $\chi_\Pi(e) = \dim V$
- $\chi_{\Pi \oplus \Sigma} = \chi_\Pi + \chi_\Sigma$
- $\chi_{\Pi^\vee} = \overline{\chi_\Pi}$
- $\chi_{\Pi \otimes \Sigma} = \chi_\Pi \cdot \chi_\Sigma$
- $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_\Pi}\chi_\Sigma$
- $\chi_\Pi(hgh^{-1}) = \chi_\Pi(g)$

The last condition says that characters are invariant under conjugation.

1.3. **Class functions.** Functions $f \colon G \to \mathbf{C}$ invariant under conjugation play a central role, we call them *class functions.* They are the same thing as functions on G/G, where G is acting on itself by conjugation. We write $\mathbf{C}[G/G]$ for the vector space of class functions. We may define an inner product on $\mathbf{C}[G/G]$ as follows.

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} \overline{f(x)} g(x).$$

1.4. **Projection operator.** Any G-rep V has a subrepresentation

$$V^G = \{v \in V \mid gv = v, \forall g \in G\}$$

called the subspace of *invariants.* If $\mathbf{C}$ denotes the trivial one-dimensional representation of G, then $V^G = \mathrm{Hom}_G(\mathbf{C}, V)$.

Proposition 1.6. Let V be a G-rep. Let $P \colon V \to V$ be the operator defined by

$$Pv = \frac{1}{|G|} \sum_{g \in G} g \cdot v.$$

Then,

- $P^2 = P$
- $\mathrm{Im}\, P = V^G$
- $P|V^G = \mathrm{id}_{V^G}$.

Using this, we deduce a key result.

Theorem 1.7. Let V, W be two representations. Then

$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W).$$

Proof. Let Γ denote the representation $\mathrm{Hom}(\mathrm{V}, \mathrm{W})$.

$$\begin{aligned}
\langle \chi_{\mathrm{V}}, \chi_{\mathrm{W}} \rangle &= \frac{1}{|\mathrm{G}|} \sum_{x \in \mathrm{G}} \overline{\chi_{\mathrm{V}}(x)} \chi_{\mathrm{W}}(x) \\
&= \frac{1}{|\mathrm{G}|} \sum_{x \in \mathrm{G}} \chi_{\mathrm{V}^\vee \otimes \mathrm{W}}(x) \\
&= \frac{1}{|\mathrm{G}|} \sum_{x \in \mathrm{G}} \chi_{\mathrm{Hom}(\mathrm{V},\mathrm{W})}(x) \\
&= \frac{1}{|\mathrm{G}|} \sum_{x \in \mathrm{G}} \mathrm{tr}(\Gamma(x)) \\
&= \mathrm{tr}\left(\sum_{x \in \mathrm{G}} \frac{1}{|\mathrm{G}|} \Gamma(x) \right) \\
&= \mathrm{tr}(\mathrm{P}) \\
&= \dim \mathrm{Im}\, \mathrm{P} \\
&= \dim \mathrm{Hom}_{\mathrm{G}}(\mathrm{V}, \mathrm{W}).
\end{aligned}$$

□

Combining this, once again, with Maschke's theorem and Schur's lemma tells us that characters of irreps form an orthonormal system in $\mathbf{C}[\mathrm{G}/\mathrm{G}]$. In particular:

Two irreps are isomorphic if and only if they have the same character.

But even more is true!

Theorem 1.8. Let $\mathrm{V}_1, \ldots, \mathrm{V}_k$ be all the distinct G-irreps, and let χ_i be the corresponding characters. Then

- the χ_i are orthonormal:
$$\langle \chi_i, \chi_j \rangle = \delta_{ij}.$$
- $\chi_1, \ldots, \chi_k$ form a basis for $\mathbf{C}[\mathrm{G}/\mathrm{G}]$.

As a consequence, the number if G-irreps is equal to $|\mathrm{G}/\mathrm{G}|$, the number of conjugacy classes of G.

Proof. We already know that $\chi_1, \ldots, \chi_k$ are orthonormal. So we need to show they span the space of class functions. Suppose $f\colon \mathrm{G} \to \mathbf{C}$ is a class function, orthogonal to all the characters. We want to show that $f = 0$. Let $\gamma \in \mathbf{C}[\mathrm{G}]$ be defined as

$$\gamma = \sum_{x \in \mathrm{G}} \overline{f(x^{-1})} x$$

Two things: $\gamma \in \mathrm{Z}(\mathbf{C}[\mathrm{G}])$ belongs to the centre; $\gamma = 0$ if and only if $f = 0$. Now, $\gamma = 0$ if and only if the multiplication by γ map $m_\gamma\colon \mathbf{C}[\mathrm{G}] \to \mathbf{C}[\mathrm{G}]$ is zero.

Since γ is central, m_γ is $\mathbf{C}[\mathrm{G}]$-linear. In particular, it preserves any submodule of $\mathbf{C}[\mathrm{G}]$. As $\mathbf{C}[\mathrm{G}]$ decomposes (as a module over itself) into irreps, we deduce that $f = 0$ if and only if for any irrep V the G-equivariant map

$$\begin{aligned}
\phi\colon \mathrm{V} &\to \mathrm{V} \\
v &\mapsto \sum_{x \in \mathrm{G}} \overline{f(x^{-1})} g \cdot v
\end{aligned}$$

is zero. By Schur's lemma, this map is given by multiplication by a scalar λ. Hence

$$(\dim \mathrm{V})\lambda = \operatorname{tr}\phi = |\mathrm{G}|\langle f, \chi\rangle$$

where χ is the character of V. But f is orthogonal to all characters of irreps, hence $\lambda = 0$, hence $f = 0$. □

2. Lie groups

It turns out that many things may be generalized for compact groups. Even more pleasantly, the results come together with the representation theory of semisimple Lie algebras. Let now G be a *connected* Lie group.

Theorem 2.1. Let G be a connected Lie group. There exists a compact subgroup $\mathrm{K} < \mathrm{G}$, maximal with respect to inclusion. Moreover, if $\mathrm{K}, \mathrm{K}' < \mathrm{G}$ are maximal compact subgroups, then K, K' are conjugate. Finally, if K is a maximal compact subgroup, then G is diffeomorphic to $\mathrm{K} \times \mathbf{R}^d$.

Noncompact groups are hard, so let's restrict to compact ones. Let us fix a *compact, connected* Lie group K. We wish to study finite-dimensional complex representations.

Theorem 2.2 (Weyl's unitary trick). Reps of K are completely reducible.

Proposition 2.3 (Schur's lemma). If V, W are K-irreps, then $\mathrm{Hom}_{\mathrm{K}}(\mathrm{V}, \mathrm{W})$ is either zero or isomorphic to $\mathbf{C}$.

At this point, we would want an analogue of the group algebra, or at the very least a nice space of class functions. Defining naively $\mathbf{C}[\mathrm{K}]$ to be functions on K leads to problems, as K is infinite. We could define it to be functions with compact support. However, this would treat K as a *discrete* group, but we want to remember the topology of K (for example if we wish to use Lie algebras later on).

2.1. Haar measure.

What we need is a function space which remembers the topology of K. The first thing which comes to mind is $\mathscr{C}(\mathrm{G})$ (or $\mathscr{C}^\infty(\mathrm{K})$) the space of continuous (or smooth) functions $\mathrm{K} \to \mathbf{C}$. What was really neat in the finite group case was the orthonormality of characters. So we need an inner product. Let dx be the *Haar measure* on K. This is a left-invariant measure such that vol(K) $= 1$. To construct it, you pick arbitrarily a top form on $\mathfrak{k}$, the Lie algebra of K, and then left translate it to obtain a volume form on K. Since K is compact, it will have finite volume. We normalize by dividing out by the volume, obtaining the Haar measure.

As usual, we may define an inner product as

$$\langle f, g\rangle = \int_{\mathrm{K}} \overline{f(x)} g(x)\, dx.$$

Since we now have a measure on a compact space, the reasonable thing to do would be to look at $\mathrm{L}^2(\mathrm{K})$. Indeed, the Peter-Weyl theorem proves (among other things) that $\mathrm{L}^2(\mathrm{K})$ splits off (as a K-representation) as a (completed) direct sum of irreps. Moreover, just like for $\mathbf{C}[\mathrm{G}]$, all irreps appear in this decomposition.

3. Real roots and weights

A *torus* is a Lie group T isomorphic (as a Lie group) to $(\mathrm{S}^1)^k$. Crucial in the theory of Lie algebras of compact groups is finding a torus $\mathrm{T} < \mathrm{K}$, maximal with respect to inclusion.

Example 3.1. In $\mathrm{U}(n)$ a maximal torus is given by diagonal matrices (in $\mathrm{U}(n)$). Explicitly, the diagonal entries are of the form $(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_{n-1}}, e^{-i(\theta_1+\cdots+\theta_{n-1})})$

Theorem 3.2. Let K be compact and connected.

- There exists $\mathrm{T} < \mathrm{K}$ a torus, and maximal with respect to inclusion.
- Any two maximal tori $\mathrm{T}, \mathrm{T}' < \mathrm{K}$ are conjugate.
- If $x \in \mathrm{K}$, there exists a maximal torus $\mathrm{T} < \mathrm{K}$ with $x \in \mathrm{T}$.
- $\exp\colon \mathfrak{k} \to \mathrm{K}$ is surjective
- $x \in \mathrm{K}$ is in the centre of K if and only if it belongs to all maximal tori in K.

3.1. **Weyl group.** So, let us fix K, its Lie algebra $\mathfrak{k}$, a maximal torus $\mathrm{T} \subset \mathrm{K}$, and the corresponding subalgebra $\mathfrak{t} \subset \mathfrak{k}$. Since K is compact, we may fix an Ad-invariant inner product on $\mathfrak{k}$.

Definition 3.3. The *Weyl group* of K is the group $\mathrm{W} = \mathrm{N(T)/T}$, where $\mathrm{N(T)}$ is the normalizer of T in K, i.e.

$$\mathrm{N(T)} = \{x \in \mathrm{K} \mid x\mathrm{T}x^{-1} = \mathrm{T}\}$$

is the largest subgroup in which T is normal.

The Weyl group acts on $\mathfrak{t}$ via the adjoint action.

3.2. **Roots.** When K is simply-connected, the complexification $\mathfrak{g} = \mathfrak{k}_{\mathbf{C}}$ is semisimple. The representation theory of $\mathfrak{g}$ can then be studied using weights and roots. Recall that $\mathfrak{h} = \mathfrak{t}_{\mathbf{C}}$ is a Cartan subalgebra and that (using our inner product to identify $\mathfrak{g}$ with $\mathfrak{g}^{\vee}$) weights and roots live in $i\mathfrak{t}$.

When K is not simply-connected, $\mathfrak{g}$ is not semisimple. However, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}$, with $\mathfrak{z}$ the centre and $(\mathfrak{k}_1)_{\mathbf{C}}$ semisimple. In this context, one can still speak of roots and weights. For groups, the convention is to speak of *real* roots and weights, which live in $\mathfrak{t}$ rather than $i\mathfrak{t}$.

Definition 3.4. $\alpha \in \mathfrak{t}$ is a *real root* if there is a non-zero $\mathrm{X} \in \mathfrak{g}$ such that

$$[\mathrm{H}, \mathrm{X}] = i\langle \alpha, \mathrm{H}\rangle \mathrm{X}$$

for all $\mathrm{H} \in \mathfrak{t}$.

The roots of K form a root system, with the exception that they won't span $\mathfrak{t}$, but only $\mathfrak{t}_1$. Let us fix a base Δ for the real roots. As usual, we define $\lambda \in \mathfrak{t}$ to be *algebraically integral* if

$$2\frac{\langle \lambda, \alpha\rangle}{\alpha, \alpha} \in \mathbf{Z}$$

for all real roots α; *dominant* if

$$\langle \lambda, \alpha \rangle \geq 0$$

for all $\alpha \in \Delta$. Using Δ, we can also define when one element is *higher* than another.

The new feature one needs to take into account is the notion of *analytically* integral element. Let

$$\Gamma = \{\mathrm{H} \in \mathfrak{t} \mid \exp(2\pi \mathrm{H}) = e\}$$

be the "kernel" of the exponential map for $\mathfrak{t}$.

Definition 3.5. We say $\lambda \in \mathfrak{t}$ is *analytically integral* if

$$\langle \lambda, \mathrm{H} \rangle \in \mathbf{Z}$$

for all $\mathrm{H} \in \Gamma$.

If α is a real root, we define the corresponding real coroot to be

$$\mathrm{H}_\alpha = 2\frac{\alpha}{\langle \alpha, \alpha \rangle}.$$

Theorem 3.6. Let α be a real root. Then $\mathrm{H}_\alpha \in \Gamma$.
Moreover,

analytically integral implies *algebraically integral*.

I.e. if μ is an analytically integral element then μ is algebraically integral.
If K is simply-connected, then algebraically integral elements are also analytically integral.

For example, for $\mathrm{SO}(3)$ the algebraically integral elements may be identified with $\frac{1}{2}\mathbf{Z}$, while the analytically integral elements with $\mathbf{Z}$. While for $\mathrm{SU}(2)$ the analytic and algebraic are the same.

3.3. **Weights.** Given a representation (Π, V) of K, with associated representation (π, V) of $\mathfrak{g}$, we define $\lambda \in \mathfrak{t}$ to be a *real weight* if there is $0 \neq v \in \mathrm{V}$ such that

$$(\pi \mathrm{H})v = i\langle \lambda, \mathrm{H} \rangle v$$

for all $\mathrm{H} \in \mathfrak{t}$.

Just as for Lie algebras, the goal is to prove a theorem of the highest weight.

Theorem 3.7. Let K be a compact, connected Lie group, with maximal torus T and a choice of a base Δ.

(1) Every irrep has a highest weight.
(2) Two irreps are isomorphic if and only if they have the same highest weight.
(3) The highest weight of an irrep is a dominant integral element.
(4) Given any dominant and *analytically* integral element μ. There exists an irrep with highest weight μ.

Parts 1 and 3 follow as a K-irrep induces a $\mathfrak{g}$-irrep, and we use the theorem of the highest weight for those. Parts 2 and 4, however, have a different flavor.

3.4. **Characters again.** We will show that Theorems 1.5, 1.7, 1.8 have analogues for compact groups. This will actually allow us to deduce the theorem of the highest weight.
The *character* of a K-rep (Π, V) is defined to be

$$\begin{aligned} \chi_\Pi \colon \mathrm{K} &\to \mathbf{C} \\ x &\mapsto \mathrm{tr}(\Pi(x)) \end{aligned}$$

and notice that if $\mathrm{X} \in \mathfrak{k}$ then

$$\chi_\pi(\mathrm{X}) = \mathrm{tr}(e^{(\pi \mathrm{X})}) = \chi_\Pi(e^{(\pi \mathrm{X})}).$$

Recall that characters are *class functions*, i.e. functions invariant under conjugation. Indeed, if $x, y \in \mathrm{K}$ then

$$\chi_\Pi(xyx^{-1}) = \mathrm{tr}(\Pi(x)\Pi(y)\Pi(y)^{-1}) = \chi_\Pi(y).$$

The space of (continuous) class functions will be operate as a deus ex machina.

Theorem 3.8. The formulae from Theorem 1.5 hold verbatim for compact groups. If $(\Pi, V), (\Sigma, W)$ are two reps, then

$$\langle \chi_V, \chi_W \rangle = \int_K \overline{\operatorname{tr}(\Pi(x))} \operatorname{tr}(\Sigma(x))\, dx = \operatorname{Hom}_K(V, W).$$

In particular, if Π and Σ are K-irreps, then

$$\int_K \overline{\chi_\Pi(x)} \chi_\Sigma(x)\, dx = \begin{cases} 1 & \text{if } \Pi \simeq \Sigma \\ 0 & \text{otherwise} \end{cases}$$

The proof is actually exactly the same as for finite groups, as for any rep V we may define the averaging operator P by

$$Pv = \int_K \Pi(x) v dx$$

as a V-valued (or Bochner, if you want to be fancy) integral.

3.5. **Weyl character formula**. So, given complete reducibility and Schur's lemma, we see that the character of an irrep is a complete invariant. Part (2) of Theorem 3.7 actually follows from this, after the sledgehammer that is Weyl's character formula.

Theorem 3.9 (Weyl's character formula). Suppose (Π, V) is an irrep of K with highest weight μ. We have

$$q(H)\chi_\Pi(e^H) = \sum_{w \in W} \det(w) \exp\left(i \langle w \cdot (\mu + \delta), H \rangle\right)$$

where W is the Weyl group (with respect to our chosen maximal torus), δ is half the sum of the positive roots, $H \in \mathfrak{t}$ and $q(H)$ is the *Weyl denominator*

$$q(H) = \sum_{w \in W} \det(w) \exp\left(\langle w \cdot \delta, H \rangle\right)$$

Since we are skipping (all) details, the explicit formula isn't really the point. What is to notice is that the expression for the character depends on the highest weight μ. Therefore, if two irreps have the same highest weight, they have the same character, which means they must be isomorphic.

3.6. **Peter-Weyl**. The true miracle is the analogue of Theorem 1.8.

Theorem 3.10 (Completeness of Characters). Let f be a continuous class function such that

$$\int_K \overline{f(x)} \chi_\Pi(x)\, dx = 0$$

for all irreps Π. Then $f \equiv 0$.

This result is part of the jackhammer that is the *Peter-Weyl* theorem.

3.7. **Fourier theory**. Let us say only a couple of words of how Peter-Weyl relates to Fourier theory. Let K be a compact group and let $\hat{\mathrm{K}}$ be the set of irreducible characters (i.e. characters of irreducible representations). Instead of looking directly at class functions, Peter-Weyl proves something stronger (and then deduces the relationship between class functions and characters as a consequence). Let $\mathrm{L}^2(\mathrm{K})$ be the Hilbert space of square-integrable functions $f\colon \mathrm{K} \to \mathbf{C}$, with respect to the Haar measure on K. This space is a (unitary) K-representation of K, by declaring

$$(g \cdot f)(x) = f(g^{-1}x).$$

Peter-Weyl shows (among other things), that

$$\mathrm{L}^2(\mathrm{K}) = \widehat{\bigoplus_{\mathrm{V}\in\hat{\mathrm{K}}}} \mathrm{V}^{m_{\mathrm{V}}}$$

i.e. the L^2 space is the (completed) direct sum the all irreps V of K, each appearing with multiplicity $m_{\mathrm{V}} = \dim \mathrm{V}$.

The way we should interpet the symbol $\mathrm{W} = \widehat{\bigoplus_i} \mathrm{W}_i$ is that any vector $w \in \mathrm{W}$ may be ***uniquely*** written as the sum of as series $w = \sum_i w_i$ with $w_i \in \mathrm{W}_i$.

Let us specialize to the case $\mathrm{K} = \mathrm{S}^1$. One shows that, since S^1 is abelian, its irreps are all 1-dimensional. Moreover, they (or, equivalently, their characters) are all of the form $\theta \mapsto e^{in\theta}$. So $\widehat{\mathrm{S}^1} = \mathbf{Z}$. [Since S^1 is abelian (and locally compact), $\widehat{\mathrm{S}^1}$ is also a group, called the ***Pontryagin dual***.]

But $\mathrm{L}^2(\mathrm{S}^1)$ is the same as periodic (of period 2π), complex-valued, square-integrable functions on $\mathbf{R}$. Fourier's classic theorem tells us that, if $f \in \mathrm{L}^2(\mathrm{S}^1)$ then

$$f(\theta) = \sum_{n\in\mathbf{Z}} a_n e^{in\theta}$$

where

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{-in\theta}\, d\theta.$$

So, let V_n denote the irrep of S^1 where $\mathrm{V}_n = \mathbf{C}$ as a vector space and $\theta \cdot z = e^{in\theta} z$. Fourier's theorem is really telling us that

$$\mathrm{L}^2(\mathrm{S}^1) = \widehat{\bigoplus_{n\in\mathbf{Z}}} \mathrm{V}_n$$

and remember that $\mathbf{Z} = \widehat{\mathrm{S}^1}$.

Indeed, $\theta_0 \in \mathrm{S}^1$ acts on $\mathrm{L}^2(\mathrm{S}^1)$ by taking $f(\theta)$ to $f(\theta - \theta_0)$. We may view the vector space V_n as being spanned by $e^{-in\theta}$, so that S^1 indeed acts correctly on it:

$$\theta_0 \cdot (e^{-in\theta}) = e^{-in(\theta-\theta_0)} = e^{in\theta_0} e^{-in\theta}.$$

There are certainly uncountably many more things to say about this story, but we will not comment on this any further.

3.8. **Highest weight again**. A non-constructive proof of part (4) Theorem 3.7 may be given as follows. Let μ be a dominant, analytically integral element. How do we show a corresponding K-rep V_μ exists? One trick is to prove the following string of non-trivial results.

Let μ be a dominant analytically integral element. Weyl's character formula tells us at least what the character of the rep we wish to construct should look like.

Proposition 3.11. There is a unique continuous function $\phi_\mu \colon T \to \mathbf{C}$ such that

$$q(H)\phi_\mu(e^H) = \sum_{w\in W} \det(w) e^{i\langle w\cdot(\mu+\delta), H\rangle}$$

for all $H \in \mathfrak{t}$.

Proposition 3.12. Let $f \colon T \to \mathbf{C}$ be continuous and invariant under the action of the Weyl group on T. Then there exists a unique class function $F \colon K \to \mathbf{C}$, with $F|T = f$.

Call now Φ_μ the extension of ϕ_μ to K.

Proposition 3.13. The collection $\{\Phi_\mu\}_\mu$ where μ ranges among all dominant analytically integral elements is an orthonormal system of class functions.

With all this, part (4) of Theorem 3.7 actually becomes provable.

Proof. If V is a K-irrep with highest weight μ, Weyl's character formula tells us that $\Phi_\mu = \chi_V$. Suppose there a dominant analytically integral element μ_0 which does not correspond to a K-irrep. The function Φ_{μ_0} still makes sense, and is orthogonal to Φ_μ for all other μ. In particular, Φ_{μ_0} is orthogonal to χ_V for all K-irreps V. By Peter-Weyl, $\Phi_{\mu_0} = 0$, which is a contradiction. □

4. Fundamental Groups

Let us summarize the results from the very last chapter of Hall. Using the long exact sequence for a fibration, and the fact that $\pi_i(S^n) = 0$ for $i < n$, one can deduce some basic facts.

Theorem 4.1. We have $SO(n)/SO(n-1) = S^{n-1}$, so $\pi_1(SO(n)) = \mathbf{Z}/2\mathbf{Z}$ for $n \geq 3$.

Theorem 4.2. We have $SU(n)/SU(n-1) = S^{2n-1}$. So $\pi_1 SU(n) = 1$ for all $n \geq 2$.
On the other hand, $\pi_1 U(n) = \mathbf{Z}$ for all $n \geq 1$.

Theorem 4.3. For the compact symplectic group $Sp(n) = Sp(n, \mathbf{C}) \cap U(2n)$, we have $Sp(n)/Sp(n-1) = S^{4n-1}$. We deduce that $\pi_1(Sp(n)) = 1$ for all $n \geq 1$.

Polar decomposition lets also say things about noncompact groups.

Theorem 4.4. We have $\pi_1(GL(n, \mathbf{C})) = \pi_1(U(n)) = \mathbf{Z}$, for all $n \geq 1$.
We have $\pi_1(SL(n, \mathbf{C})) = 1$, for all $n \geq 2$.
We have $\pi_1(SL(n, \mathbf{R})) = \pi_1(SO(n)) = \mathbf{Z}/2\mathbf{Z}$, for all $n \geq 2$.
On the other hand, $\pi_1(SL(2, \mathbf{R})) = \pi_1(SO(2)) = \mathbf{Z}$.

4.1. **General results**. Pick now a maximal torus $T < K$ of our compact connected Lie group.

Proposition 4.5. Any loop in K is homotopy to a loop in T.

Notice that a loop in T may define a nonzero element of $\pi_1(T)$ but be zero in $\pi_1(K)$. Recall we define $\Gamma < \mathfrak{t}$ to be the kernel of $e^{2\pi\cdot}$.

Theorem 4.6. $\pi_1(\mathrm{T})$ is isomorphic to Γ via the map sending $\gamma \mapsto e^{2\pi\gamma t}$ with $t \in [0,1]$.

Recall we have $\mathrm{R} \subset \mathfrak{t}$, the set of real roots.

Definition 4.7. The *coroot lattice* is the subset $\mathrm{I} \subset \mathfrak{t}$ consisting of **Z**-linear combinations of the coroots H_α, with $\alpha \in \mathrm{R}$.

Theorem 4.8. For each $\gamma \in \Gamma$, the loop $t \mapsto e^{2\pi\gamma t}$ is nullhomotopic *in* K if and only if $\gamma \in \mathrm{I}$.

Corollary 4.9. $\pi_1(\mathrm{K}) = \Gamma/\mathrm{I}$.