Gabriel's Theorem and Points

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This is really fun to state.

> "A variety is completely recovered by its category of coherent sheaves."

Let's be a bit more precise about it.

Theorem (Gabriel, Rosenberg, Gabber/Brandenburg) – Let X, Y be two quasi-separated schemes, then

— $\mathrm{qc}(X) \simeq \mathrm{qc}(Y)$

if and only if

— $X \simeq Y$

(when X and Y are noetherian you can replace qc with Coh).[1] When working over a ground ring $\mathbb{C}$ then qc is moreover a $\mathbb{C}$-linear category and we can dress up the theorem by saying $\mathrm{qc}(X)$ as a $\mathbb{C}$-linear category recovers X as a $\mathbb{C}$-scheme. *

The whole project started out from this question: wouldn't it be cool to prove this theorem by realising X as a moduli space of "points" of $\mathrm{Coh}(X)$?
Anyway, before answering this question let's say some other stuff.

Affine Case — For affine schemes proving the reconstruction theorem is actually incredibly straightforward. Given a category C there is the so-called *centre*

$$Z(C) = \mathrm{Nat}(1_C, 1_C)$$

which is by definition the monoid of natural transformations of the identity functor. When C happens to be additive (which is an *intrinsic* property of the category!) then $Z(C)$ is a commutative ring. For any ring R you can check that

$$Z(R\text{-}\mathrm{Mod}) = Z(R)$$

the centre of its category of modules is actually the centre of the ring.
Therefore, for an affine scheme X, it follows that

$$X = \mathrm{Spec}\, Z(\mathrm{qc}(X)).$$

How can we extend this to schemes?

[1] By $\mathrm{qc}(X)$ I mean the category of quasi-coherent modules on X.

Schemes — OK, I hope you can forgive this mildly patronizing section. Well, what is a scheme? There are at least two ways to present a scheme.

1. X = Zariski topological space + sheaf of rings = ringed space $(|X|, \mathcal{O}_X)$.
2. X = moduli space = collection of all maps $S \to X$, for $S \in \mathtt{Aff}$

$$X = \text{“}\{S \to X \mid S \in \mathtt{Aff}\}\text{”}$$

To be more pedantic,

— X = sets $\mathrm{Hom}(S, X)$ + for each $T \to S$, a function $\mathrm{Hom}(S, X) \to \mathrm{Hom}(T, X)$.

In other words X is the *functor* $\mathrm{Hom}(-, X)$.

In spite of looking tautological and useless, we like approach 2. better.[2] Things like M_g (the moduli space of curves) or Hilb_X (the Hilbert scheme) and, why not, even $\mathbb{P}^n$ or Grassmannians are naturally moduli spaces.

Topological Strategy — In any case, the Gabriel-Rosenberg way to prove the reconstruction theorem goes via 1. Let's assume everything is noetherian just to be safe. First, a definition.

1.1 Definition – Let A be an abelian category[3]. A *Serre* subcategory of A is a subcategory $S \subset A$ such that, for all short exact sequences

$$0 \to A \to B \to C \to 0$$

then $A, C \in S$ if and only if $B \in S$. *

With an abelian category A we associate a ringed space $(|A|, \mathcal{O}_A)$. The idea is the following. There is a bijection

$$\{\text{irreducible closed subsets of } |X|\} \leftrightarrow \{\text{Serre subcategories of } \mathtt{Coh}(X)\}.$$

The set on the right is defined purely in terms of the category theory of $\mathtt{Coh}(X)$, so it is invariant under an equivalence $\mathtt{Coh}(X) \simeq \mathtt{Coh}(Y)$. One direction is easy

$$(Y \subset |X|) \mapsto \mathtt{Coh}_Y X = \{F \in \mathtt{Coh}(Y) \mid \mathrm{supp}\, F \subset Y\}.$$

The hard part is going back.

[2] This might be called the Yoneda or Grothendieck functor-of-points view.
[3] This means it's a category where we can take kernels, cokernels, sums and pretend we are in $R\text{-}\mathtt{Mod}$ without feeling bad about it.

Extending the Theorem — A first question comes to mind: can the theorem be strengthened? Notice that

$$\mathtt{qc}(B\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} - \mathtt{Vect} = \mathtt{qc}(\{\bullet, \bullet\}).$$

The first is modules over the classifying stack of $\mathbb{Z}/2\mathbb{Z}$ (which is just representations of $\mathbb{Z}/2\mathbb{Z}$), the third is quasi-coherent sheaves over two points. Both categories are equal to that of super (i.e. $\mathbb{Z}/2\mathbb{Z}$-graded) vector spaces. This shows that the theorem fails terribly for algebraic stacks.[4]

Sitting in between schemes and stacks are algebraic spaces. Schemes are obtained by gluing affine schemes along Zariski open immersions. If we allow the gluings to be along ètale maps, we get algebraic spaces. This allows for more flexibility.

— If $\mathtt{G}$ is acting *freely* on $\mathtt{X}$ then the quotient $\mathtt{X}/\mathtt{G}$ doesn't always exist in the category of schemes. It does, however, in the category of algebraic spaces! This result is not super well-known, I guess it's because the interesting group actions are rarely free and that's where you need stacks.

— Also, $\mathrm{Hilb}_{\mathtt{X}}$ for $\mathtt{X}$ projective is projective but Hironaka showed that there are cases where $\mathtt{X}$ is smooth and proper but $\mathrm{Hilb}_{\mathtt{X}}$ is ***not*** a scheme, just an algebraic space.

Algebraic spaces *cannot* be described as ringed spaces (ringed topoi maybe, but who wants to work with those?) so a modular approach is really needed.

Moduli of Points — The idea is the following: given an abelian category $\mathtt{A}$ we construct a moduli-space (i.e. a functor) $\mathtt{P}_{\mathtt{A}}$ of "points" of $\mathtt{A}$. This is a purely categorical construction, so that if $\mathtt{A} \simeq \mathtt{B}$ then $\mathtt{P}_{\mathtt{A}} \simeq \mathtt{P}_{\mathtt{B}}$.

The main point is that $\mathtt{P}_{\mathtt{qc(X)}} = \mathtt{X}$ when $\mathtt{X}$ is a quasi-compact and separated algebraic space.

1.2 Theorem (myself-Grochenig) – Let $\mathbb{C}$ be a ground ring. Let $\mathtt{X}, \mathtt{Y}$ be quasi-compact and separated algebraic spaces over $\mathbb{C}$. Then

— $\mathtt{qc}(\mathtt{X}) \simeq \mathtt{qc}(\mathtt{Y})$ as $\mathbb{C}$-linear categories

if and only if

— $\mathtt{X} \simeq \mathtt{Y}$ as spaces over $\mathbb{C}$.

Moreover, if $\mathtt{X}$ or $\mathtt{Y}$ is flat over $\mathbb{C}$, then

— $\mathrm{Aut}_{\mathbb{C}}(\mathtt{qc}(\mathtt{X})) \simeq \mathrm{Aut}_{\mathbb{C}}(\mathtt{X}) \ltimes \mathrm{Pic}(\mathtt{X})$

in other words all $\mathbb{C}$-linear autoequivalences of $\mathtt{qc}(\mathtt{X})$ are of the form $\mathtt{f}^*(-) \otimes \mathtt{L}$ for $\mathtt{f}$ an automorphism of $\mathtt{X}$ over $\mathbb{C}$ and $\mathtt{L} \in \mathrm{Pic}(\mathtt{X})$.[5] ⋆

[4] To make it work you have to consider $(\mathtt{qc}(\mathtt{X}), \otimes)$ as a *tensor* category (work of Balmer, Lurie, Brandenburg).

[5] It's important to point out that Brandenburg was able to prove this result for schemes without any flatness assumptions. Also, for smooth projective varieties this was known for a long time.

While we are at it, we mention yet another generalization of this.

1.3 Theorem (Perego, Canonaco-Stellari, Antieau) – Let X, Y be a quasi-separated schemes over $\mathbb{C}$ and let α, β be two Brauer classes. Then $\mathrm{qc}(X, \alpha) \simeq \mathrm{qc}(Y, \beta)$ if and only if $X \simeq Y$ and β pulls back to α. $\ast$

I lied a little bit earlier. The moduli of points $P_{\mathrm{qc}(X)}$ is not equal to X but it is equal to the trivial gerbe $X \times B\mathbb{G}_m$. In general, it turns out that if α is *any* class in $H^2(X, \mathcal{O}_X^\times)$ then $P_{\mathrm{qc}(X,\alpha)} = \alpha$, i.e. the gerbe corresponding to α.

1.4 Theorem (myself-Groechenig) – Let X, Y be two quasi-compact and separated algebraic spaces over $\mathbb{C}$. Let α, β be two classes in $H^2(\mathcal{O}^\times)$ (not necessarily coming from an[6] Azumaya algebra). Then $\mathrm{qc}(X, \alpha) \simeq \mathrm{qc}(Y, \beta)$ if and only if $X \simeq Y$ and β pulls back to α. $\ast$

Sketch — The idea is that we want to recover X as a moduli space of points in $\mathrm{qc}(X)$. What is X? It's the datum of all maps $S \to X$. How does this relate to $\mathrm{qc}(X)$? Well, given a map $f\colon S \to X$ we can take the structure sheaf of the graph $\mathcal{O}_\Gamma$, which is an element of $\mathrm{qc}(S \times X)$.[7] We should think of $\mathcal{O}_\Gamma$ as a *family* of points of X parameterized by S. What are the properties enjoyed by $\mathcal{O}_\Gamma$? Let's write down a diagram.

$$\begin{array}{ccc} S \times X & \xrightarrow{q} & X \\ \downarrow{\scriptstyle p} & & \\ S & & \end{array}$$

— $\mathcal{O}_\Gamma$ is flat over S

— the functor $p^*(-) \otimes \mathcal{O}_\Gamma$ induces a bijection between closed subschemes of S and quotients of $\mathcal{O}_\Gamma$

plus three other conditions we won't focus on: $\mathcal{O}_\Gamma$ is finitely generated, the functor $p^*(-) \otimes \mathcal{O}_\Gamma$ is fully-faithful and the functor $\mathrm{RHom}(- \otimes^L \mathcal{O}_S, \mathcal{O}_\Gamma)$ sends compact objects to compact objects.

When X is separated, a sheaf $P \in \mathrm{qc}(S \times X)$ is the structure sheaf of a graph if and only if it satisfies those abstract properties. Given an abelian category, we are ready to define our functor.

1.5 Definition – Let A be a (Grothendieck) abelian category. We define a moduli functor as follows. With each $S = \mathrm{Spec}\, R$ we associate $P_A(S)$ given by those $Q \in A \otimes R$ such that

— Q is flat over R,

— Q the functor $P \otimes_R (-)$ induces a bijection between closed subschemes of S with quotients of P

[6] Or possibly a derived one, in Antieau's case.
[7] Here is where separatedness creeps in, to ensure $\mathcal{O}_\Gamma$ is equivalent to f.

plus the other properties. ∗

The base-changed category $A \otimes R$ is easy to define and comes with an action of R-Mod, in the sense that for any module M and object P the tensor product $P \otimes_R M$ makes sense. In the geometric case $qc(X) \otimes R = qc(S \times X)$ and $P \otimes_R M = P \otimes_{S \times X} p^* M$.

Something Else — I'll finish by saying something vague. The group $H^0(X, \mathcal{O}_X^\times)$ classifies invertible functions. The group $H^1(X, \mathcal{O}_X^\times)$ classifies invertible modules, i.e. line bundles. Another way of saying this is that it classifies $\mathbb{G}_m$-torsors. For $H^2(X, \mathcal{O}_X^\times)$, we certainly have the description in terms of $B\mathbb{G}_m$-torsors. But there's also a linear description as well, similar to that of vector bundles. We call a sheaf of abelian categories over X *invertible* if locally on opens U_i it is isomorphic to $qc(U_i)$. It turns out that H^2 also classifies invertible sheaves of abelian categories.

What about H^3?