SHEAVES4KNOTS

by

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Abstract. — Notes for a talk given at the topology working seminar, Rice 2017-03-03.

The goal is to use a theorem of Shende in knot theory as motivation to learn some sheaf theory.

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1. Statement of the theorem

Let us start by stating the theorem as a slogan (copied from the article's abstract): knots can only have Legendrian isotopic conormal tori if they themselves are isotopic or mirror images.

1.1. Symplectic.

— If M is a manifold, a symplectic form ω is closed 2-form on M which is everywhere non-degenerate. In $\mathbb{R}^{2n}$, with coordinates $(x_1, y_1, \dots, x_n, y_n)$ the standard symplectic form is

$$\omega_{\text{std}} = \sum_i dx_i \wedge dy_i$$

Darboux's theorem then tells us that symplectic manifolds have no local invariants, as (locally) they all look the same. In other words, if (M, ω) is a symplectic manifold, then locally we may find a coordinate patch U with coordinates $(x_1, y_1, \dots, x_n, y_n)$ such that $\omega|\mathrm{U} = \sum_i dx_i \wedge dy_i$.

Associated with any manifold M there is a symplectic manifold: namely its cotagent bundle T^*M. Indeed, on T^*M there is a canonical 1-form θ called the *Liouville* form. Concretely, if $(x_1, \dots, x_n)$ are local coordinates on M and $(x_1, y_1, \dots, x_n, y_n)$ are local coordinates on T^*M, then

$$\theta = \sum_i y_i dx_i$$

and a symplectic form can be obtained by $\omega = d\theta$.

Clearly, if M and M′ are diffeomorphic, then T^*M and T^*M' are symplectomorphic (i.e. diffeomorphic via a diffeomorphism preserving the symplectic form). I believe the converse question is still open in general.

1.2. Contact. — If M is a manifold of dimension $2n+1$, a *contact structure* ξ is a sub-bundle $\xi \subset TM$ of rank $2n$, subject to the following condition. Locally, ξ can be written as $\ker \alpha$ for α a 1-form. We require this 1-form to satisfy

$$\alpha \wedge (d\alpha)^n \neq 0.$$

This condition is the extreme opposite of ξ giving rise to a foliation. In $\mathbb{R}^{2n+1}$, with coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ there is also a standard contact structure, given by the contact 1-form

$$\alpha_{\mathrm{std}} = dz + \sum_i y_i dx_i$$

which we immediately notice is the Liouville form with the extra term dz. A contact Darboux theorem also holds, stating that if (M, ξ) is a contact manifold then locally there are coordinates $(x_1, y_1, \dots, x_n, y_n, z)$ on M such that the contact 1-form becomes $dz + \sum_i y_i dx_i$.

Let M now be any manifold of dimension n. The cotangent bundle T^*M provides us with a symplectic manifold. This can be turned into a contact manifold by passing to the associated sphere bundle. In general, let E be a vector bundle on M. Then the associated sphere bundle is SE obtained as follows: take E, remove the zero section, quotient out by the scaling action of $\mathbb{R}^+$, so that $SE = (E \smallsetminus 0_E)/\mathbb{R}^+$. The fibre of ES is a sphere of dimension $r-1$ where r is the rank of E. If E has a metric, we could alternatively describe SE as the sub-bundle of unit vectors.

We call the sphere bundle of T^*M the *cosphere bundle* of M and write S^*M. One can check that the Liouville form θ induces a contact structure on S^*M. Briefly, by refining standard coordinate patches of T^*M, one may divide by the last coordinate y_n and have x_n play the role of z in α_{std}. The form just described won't be a global form, but the induced contact structure will be well defined. We should also say that a different way to get a contact manifold out of M would be to take the projectivization $\mathbb{P}T^*M$, which is $(T^*M \smallsetminus 0_{T^*M})/\mathbb{R}^*$.

1.3. Conormal torus. — Let now $N \subset M$ be a submanifold. Recall we have the normal sequence

$$0 \to TN \to TM_{|N} \to T_{N/M} \to 0$$

which is dual to the conormal sequence

$$0 \to T^*_{N/M} \to T^*M_{|N} \to T^*N \to 0.$$

If (M, ξ) is contact, we say N is *Legendrian* if $\dim M = 1 + 2\dim N$ and $TN \subset \xi|N$. So a knot $K \subset \mathbb{R}^3$ is a *Legendrian knot* if it's Legendrian with respect to the the standard contact structure on $\mathbb{R}^3$. However, we will actually be looking for Legendrians not in $\mathbb{R}^3$ but rather in $S^*\mathbb{R}^3 = \mathbb{R}^3 \times S^2$.

Once again, suppose $N \subset M$ is a submanifold. The sphere bundle associated with the conormal bundle $T^*_{N/M}$ is a submanifold of the cosphere bundle S^*M of M. Actually, more is true: just as $T^*_{N/M} \subset T^*M$ is Lagrangian, $ST^*_{N/M} \subset S^*M$ is Legendrian.

Let's specialize to the case where $M = \mathbb{R}^3$ and $N = K$ is knot, thus diffeomorphic to S^1. We call $\mathbb{T}_K$, the sphere bundle of $T^*_{K/\mathbb{R}^3}$, the *conormal torus* of the knot K in $\mathbb{R}^3$. Indeed, topologically it's just a torus. However, $\mathbb{T}_K \subset S^*\mathbb{R}^3$ is Legendrian. We can now state the theorem.

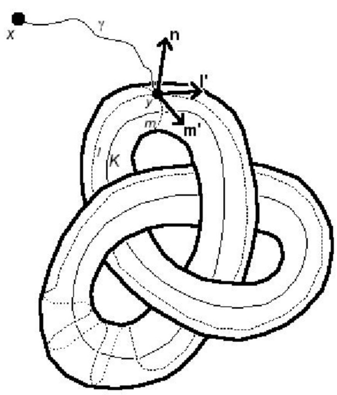
Theorem 1.1 (Shende 2016). — Let $K, K' \subset \mathbb{R}^3$ be two knots. Suppose there is a Legendrian isotopy between conormal tori $\mathbb{T}_K, \mathbb{T}_{K'} \subset S^*\mathbb{R}^3$. Then K is isotopic to either K' or its mirror image.

2. Strategy

We now sketch Shendes' strategy, but in reverse: from knot theory to sheaf theory.

2.1. Peripheral subgroups. — If K is a knot, the *knot group* is the fundamental group $\pi_1(\mathbb{R}^3 \smallsetminus K)$ of the knot complement. Now, this group does not determine the knot up to isotopy (wikipedia informs me that the square knot and the granny knot have same knot group but are not isotopic).

However, a theorem of Waldhausen says the knot group plus the extra data of its *peripheral subgroups* is a complete knot invariant. To describe the peripheral subgroups we do the following. Consider a tubular neighbourhood N of the knot. Its boundary $T = \partial N$ is a torus, hence has fundamental group $\mathbb{Z} \times \mathbb{Z}$. For generators, we can pick a 'longitude' and a 'meridian'. To do so, we first pick a point p on the boundary of T. To choose a longitude, we choose a loop with linking number zero with K. The tangent vector at p at the start of the longitude fixes an element $l \in \mathbb{R}^3$. To choose a meridian, we wrap around the boundary torus T in the orthogonal direction. A tangent vector to the meridian at p determines another vector $m \in \mathbb{R}^3$. We also have a third vector n at p, the outer normal to the boundary. We specify orientations of longitudes and meridians by requiring (m, l, n) to have the same orientation as $\mathbb{R}^3$. I stole the picture below from wikipedia (where a much better explanation of peripheral subgroup is given).



In conclusion, Waldhausen tells us that if we exhibit an isomorphism $\pi_1(\mathbb{R}^3 \smallsetminus K) \simeq \pi_1(\mathbb{R}^3 \smallsetminus K')$ which preserves the peripheral subgroups then K and K' are isotopic.

2.2. Group rings. — The next step is to linearize this fact. If G is a group, we can associate the *group ring* $\mathbb{Z}[G]$. Concretely, this can be defined as being spanned by symbols e_g for each $g \in G$ and where $e_g e_h = e_{gh}$. Any element is of the form $x = \sum_{g \in G} x_g e_g$ where $x_g \in \mathbb{Z}$ and $x_g = 0$ for all but finitely many g. We have

$$x + y = \sum_{g \in G} (x_g + y_g) e_g$$

$$xy = \sum_{g \in G} \Big(\sum_{h_1 h_2 = g} x_{h_1} x_{h_2} \Big) e_g$$

One would hope that $G \simeq G'$ as groups if and only if $\mathbb{Z}[G] \simeq \mathbb{Z}[G']$ as rings. However, this is not true even for finite groups (a counterexample was constructed by Hertwick). On the other hand, it's a conjecture of Higman that this is true when G is torsion free. He also observed that if G is left-orderable then $\mathbb{Z}[G]$ does recover G. Left orderable means there's an order $\leq$ on G, such that if $a \leq b$ then $ga \leq ga$. Thankfully, a theorem of Howie implies precisely this: knot groups are left orderable. Assembling all together we see that knowing $\mathbb{Z}[\pi(\mathbb{R}^3 \setminus K)]$, plus the data of the peripheral subgroups is enough to determine K up to isotopy.

2.3. Categories. — Now we take this to the categorical level. Instead of defining each notion one by one, let us use everything to get to the end of the proof first. We will explain some of the words appearing here later. Let M be a manifold. To M, we can attach a category $\mathtt{Sh}(M)$ of sheaves of abelian groups. Inside of it, there is a category $\mathtt{Loc}(M)$ of *local systems*. The category $\mathtt{Loc}(M)$, plus some extra data (the fibre functor), determines the group ring $\mathbb{Z}[\pi_1(M)]$. This is a version of the celebrated *Tannakian reconstruction* theorem.

To link (pun intended) this back to our theorem, we need the 'sheaf quantization theorem'.

Theorem 2.1 (Guillermou-Kashiwara-Schapira). — Suppose $\Lambda, \Lambda' \subset S^*M$ are two Legendrians in the cosphere bundle of M. A compactly supported contact isotopy on S^*M which takes Λ to Λ' induces an equivalence $\mathtt{Con}_\Lambda(M) \simeq \mathtt{Con}_{\Lambda'}(M)$.

Here $\mathtt{Con}_\Lambda(M)$ is the *constructible derived category* of sheaves on M with singular support contained in Λ.

Clearly, we want to apply this to the case of the conormal torus $\mathbb{T}_K$ of a knot. Suppose we have an Legendrian isotopy between $\mathbb{T}_K$ and $\mathbb{T}'_K$, where recall $\mathbb{T}_K \subset S^*\mathbb{R}^3$. By GKS, there is an equivalence $\phi\colon \mathtt{Con}_{\mathbb{T}_K}(\mathbb{R}^3) \to \mathtt{Con}_{\mathbb{T}_{K'}}(\mathbb{R}^3)$. The missing link is provided by Shende, who shows that the GKS equivalence actually restricts to an equivalence between $\mathtt{Loc}(\mathbb{R}^3 \setminus K) \simeq \mathtt{Loc}(\mathbb{R}^3 \setminus K')$. With some extra work, he also shows that the peripheral subgroups are preserved, up to orientation. That is why in the end he must allow K′ to be replaced by its mirror image.

3. Sheaves

OK, so what is a sheaf? It turns out that we've all known plenty of sheaves since the beginning the time.

3.1. Basics. — Let M be a manifold. If $U \subset M$ is open, we may write $\mathcal{O}_M(U)$ for the set of functions $f: U \to \mathbb{R}$ which are smooth. If $V \subset U$, we can restrict f to obtain a function $f|V: V \to \mathbb{R}$. Most importantly, $\mathcal{O}_M$ satisfies gluing: if $\{U_i\}_i$ is an open cover of U and $f_i \in \mathcal{O}_M(U_i)$ are local functions such that $f_i|U_{ij} = f_j|U_{ji}$ for all i, j, there is a unique $f \in \mathcal{O}_M(U)$ such that $f|U_i = f_i$ for all i. Here $U_{ij} = U_i \cap U_j$.

In general, a sheaf F on M consists of two pieces of data:

– an abelian group $F(U)$ for each open subset U
– a restriction map $F(U) \to F(V)$, sending f to $f|V$, for each inclusion $V \subset U$.

This data must satisfy some obvious compatibilities (e.g. if $W \subset V \subset U$, then $(f|U)|W = f|W$) and gluing, as was the case for $\mathcal{O}_M$. Elements of $F(U)$ are called the *sections* of F on U. Sheaves form a category $\mathtt{Sh}(M)$.

If $p \in M$, and $F \in \mathtt{Sh}(M)$ we can form the fibre (or, more properly, stalk) F_p. This is defined by

$$F_p = \{(f, U) | f \in F(U)\} / \sim$$

where $(f, U) \sim (g, V)$ if there exists $W \subset U \cap V$ $f|W = g|W$. We should think of F_p as *germs* of sections at p. More formally, we have defined a functor $\mathtt{Sh}(M) \to \mathtt{Mod}(\mathbb{Z})$ which takes F to F_p.

3.2. Local systems. — We've seen earlier that a special role is played by local systems, so let's define those. On M, there is a special sheaf called the *constant sheaf* $\mathbb{Z}_M$ defined by

$$\mathbb{Z}_M(U) = \{f: U \to \mathbb{Z} \mid f \text{ continuous }\}$$

in other words $\mathbb{Z}_M(U)$ is the set of locally constant functions $f: U \to \mathbb{Z}$ (because $\mathbb{Z}$ is discrete).

More generally, if A is an abelian group, we write A_M for the constant sheaf with fibre A. By definition, $A_M(U)$ is the set of locally constant functions $f: U \to A$. Notice that the fibre $(A_M)_p$ is indeed naturally isomorphic to A.

If $F \in \mathtt{Sh}(M)$ and $U \subset M$ is open, we define the restriction $F|U$ to be F but viewed as a sheaf on U.

Definition 3.1. — A sheaf $F \in \mathtt{Sh}(M)$ is called a *local system* if there is an open cover $\{U_i\}_i$ and an abelian group A, such that the restriction $F|U_i$ is isomorphic to the constant sheaf A_{U_i}. We write $\mathtt{Loc}(M) \subset \mathtt{Sh}(M)$ for the subcategory of local systems.

A local system gives rise to a *monodromy* representation. Let F be a local system. Pick a basepoint $p \in M$ and fix an isomorphism $F_p \simeq A$. Let $\gamma: [0, 1] \to M$ be a loop based at p. Pick now a trivializing cover of connected opens $\{U_i\}_i$ for F. Since we are only interested on what happens along γ, we can extract a finite collection $U_1, \dots, U_n$. Suppose we can choose these in such a way that there is a subdivision of $[0, 1]$ in subintervals $[a_i, a_{i+1}]$ and each $[a_i, a_{i+1}] \subset U_{i+1}$.

So, we start with $[a_0, a_1] = [0, a_1]$. We may trivialize $F|U_1 \simeq A$. On $[a_1, a_2]$ we may also trivialize $F|U_2 \simeq A$. Suppose the intersection $U_1 \cap U_2$ is connected. In passing from $A \to F|U_1 \to F|U_{12} \to F|U_2 \to A$ we obtain an isomorphism $\gamma_{12}: A \to A$, which of course needn't a priori be the identity. Iterating the procedure, we obtain an isomorphism $\gamma: A \to A$, by $\gamma = \gamma_{n-1n} \circ \cdots \circ \gamma_{23} \circ \gamma_{12}$. With some work, one checks that this map $A \to A$ is independente of the homotopy class of A. Hence, we just defined a representation of $\pi_1(M)$ on A! In other words, a $\mathbb{Z}[\pi_1(M)]$-module. To go in the other direction, one proceeds using the so-called *Borel construction*. It turns out the two categories are equivalent. The Tannakian theorem alluded to earlier is the fact that the actual *ring* $\mathbb{Z}[\pi(M)]$

can be recovered by $\mathtt{Loc}(M)$ together with the functor $\mathtt{Loc}(M) \to \mathtt{Mod}(\mathbb{Z})$ given by $F \mapsto F_p$ (the 'fibre' functor).

3.3. Constructible. — Slightly more general than local system, are *constructible sheaves*. Essentially, a sheaf F is constructible if there is a stratification of our space such that F restricted to each stratum is locally constant. In the case we care about, $K \subset \mathbb{R}^3$, a sheaf $F \in \mathtt{Sh}(\mathbb{R}^3)$ is *constructible* if $F|\mathbb{R}^3 \smallsetminus K$ is a local system and $F|K$ is also a local system (although we did not define what the restriction to a closed subset is).

The category $\mathtt{Con}(\mathbb{R}^3)$, called the *constructible derived category*, is then defined to be chain complexes of sheaves

$$F^\bullet : \cdots \to F^{k-1} \to F^{k+1} \to F^{k+1} \to$$

up to quasi-isomorphisms and such that each homology sheaf $H^k(F^\bullet) \in \mathtt{Sh}(\mathbb{R}^3)$ is constructible.

If F is a sheaf, or more generally an element of $\mathtt{Con}(M)$, we define its support $\operatorname{supp}(F)$ to be the set p such that $F_p \neq 0$. This is a subset of M. Notice that if $F \in \mathtt{Loc}(M)$, then $\operatorname{supp}(F) = M$.

However, in the subject of *microlocal analysis*, a more refined support is defined. The *singular support* $\operatorname{ss}(F) \subset T^*M$ (or *microsupport*) of F as the subset of $p \in M$ and co-directions $v \in T^*_pM$ such that $F_p \neq 0$ and F 'does not propagate' towards v. The precise definition is a little convoluted. We will content ourselves with pointing out that if $F \in \mathtt{Loc}(M)$, then $\operatorname{ss}(F)$ is the zero section of T^*M, as F propagates everywhere.

In general, $\operatorname{ss}(F)$ is conical, in the sense that if (p, v) belongs to it then so does (p, av) for any $a \in \mathbb{R}^+$. Hence, we can view $\operatorname{ss}(F)$ as defining a subset of the cosphere bundle S^*M. For good F, $\operatorname{ss}(F)$ is Lagrangian in T^*M (and Legendrian in S^*M). Thus, it makes sense to define a subcategory $\mathtt{Con}_\Lambda(M) \subset \mathtt{Con}_\Lambda(M)$ of constructible (complexes of) sheaves, whose singular support is a subset of Λ.

4. Going back

Let us sketch once again Shende's argument. Suppose $\mathbb{T}_K, \mathbb{T}_{K'}$, which are Legendrians in $S^*\mathbb{R}^3$ are Legendrian isotopic. By GKS, there is an equivalence $\mathtt{Con}_{\mathbb{T}_K}(\mathbb{R}^3) \simeq \mathtt{Con}_{\mathbb{T}_{K'}}(\mathbb{R}^3)$. Pick a point p, far away from either knot. By examining the effect of this functor in this special case, Shende shows that actually $\mathtt{Loc}(\mathbb{R}^3 \smallsetminus K) \simeq \mathtt{Loc}(\mathbb{R}^3 \smallsetminus K')$, in a manner compatible with taking the fibre at p. Hence, by Tannaka, we obtain an isomorphism $\mathbb{Z}[\pi_1(\mathbb{R}^3 \smallsetminus K)] \simeq \mathbb{Z}[\pi_1(\mathbb{R}^3 \smallsetminus K')]$. Since these groups are left orderable, it follows that $\pi_1(\mathbb{R}^3 \smallsetminus K) \simeq \pi_1(\mathbb{R}^3 \smallsetminus K')$. Once again, by examining closely the GKS functor, one shows that actually the peripheral subgroups are also preserved (up to some orientations). Hence, by Waldhausen, K and K′ are isotopic (or mirror images).

Now, let's go study some more sheaf theory because it's awesome.

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