The fourth homotopy group of the sphere.

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Abstract. All mistakes here are mine, you cannot have them. I'm not a topologist. By *space* I probably mean CW-complex or whatever makes the theorems below work.

I think this computation was part of Serre's thesis. It's the most amazing thing ever.

1. A computation of $\pi_4(\mathrm{S}^3)$

A map $\mathrm{E} \to \mathrm{B}$ is a fibre bundle with fibre F if locally it looks like a product. The homotopy relaxation of this concept is the notion of *fibration*. This means that for any space X and map $\mathrm{X} \times \mathrm{I} \to \mathrm{B}$ to the base, we have the so-called *homotopy lifting property*. That is, if we find a lift $\mathrm{X} \times \{0\} \to \mathrm{E}$ then we can find a lift of the whole homotopy $\mathrm{X} \times \mathrm{I} \to \mathrm{E}$.

Proposition 1.1 – The reason fibrations are the homotopy-cousin of fibre bundles is the following. Suppose $x, y \in \mathrm{B}$, then the fibres $\mathrm{E}_x, \mathrm{E}_y$ are homotopy equivalent. $*$

Thus, up to homotopy, the fibre of $\mathrm{E} \to \mathrm{B}$ stays constant.[1]

Typically we write fibrations as $\mathrm{F} \hookrightarrow \mathrm{E} \to \mathrm{B}$, where B is the *base*, E is the *total space* and F is the *fibre*. OK, here is our first piece of heavy artillery.

Theorem 1.2 – Let $\mathrm{F} \hookrightarrow \mathrm{E} \to \mathrm{B}$ be a fibration. Assume B is simply connected (more generally: $\pi_1(\mathrm{B})$ acts trivially on $\mathrm{H}^*(\mathrm{E})$). Then

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{B}; \mathrm{H}^q(\mathrm{F})) \Rightarrow \mathrm{H}^{p+q}(\mathrm{E}) \qquad *$$

When I omit coefficients of cohomology I am implicitly using $\mathbf{Z}$. [★Basically the assumptions of the theorem are saying that all the $\mathrm{R}^i f_* \mathbf{Z}_{\mathrm{E}}$ are constant sheaves (there's no twisting).★]

Remark 1.3. Assume all the $\mathrm{H}^q(\mathrm{F})$ are free abelian groups (so we don't have to deal with Tor). Then $\mathrm{H}^p(\mathrm{B}; \mathrm{H}^q(\mathrm{F})) = \mathrm{H}^p(\mathrm{B}) \otimes \mathrm{H}^q(\mathrm{F})$. In other words,

$$\mathrm{E}_2^{*,*} = \mathrm{H}^*(\mathrm{B}) \otimes \mathrm{H}^*(\mathrm{F}).$$

Let's think about the trivial case $\mathrm{E} = \mathrm{B} \times \mathrm{F}$ for a moment. Here we have Künneth telling us that in fact

$$\mathrm{H}^*(\mathrm{E}) = \mathrm{H}^*(\mathrm{B}) \otimes \mathrm{H}^*(\mathrm{F}).$$

So, in general we should think of the SSS as a big machine which takes the naive guess of what the cohomology of E should be (the E_2 page) and slowly turns it into the right answer. $*$

[1] I believe the proof of the proposition is more less the following. Take a path γ connecting x to y. Consider the map $\mathrm{E}_x \times \mathrm{I} \to \mathrm{B}$ sending (e, t) to $\gamma(t)$. We can lift this map at $t = 0$ just by $\mathrm{E}_x \to \mathrm{E}$, thus we have $\mathrm{E}_x \times \mathrm{I} \to \mathrm{E}$. Consider now $f\colon \mathrm{E}_x \to \mathrm{E}_y$ given by the inclusion of $\mathrm{E}_x \to \mathrm{E}_x \times \mathrm{I}$ at time $t = 1$. Do the same for y, obtaining $g\colon \mathrm{E}_y \to \mathrm{E}_x$. Using the lifting property again, we conclude that $g \circ f$ and $f \circ g$ are homotopic to the identity.

1.1. **Crash course on spectral sequences**. It's like reading a murder mystery where you want to guess who did it by reading as few pages as possible. The pages are boring, they consist of a grid of abelian groups, indexed by p, q (where p is the horizontal spot and q the vertical). The symbol $\mathrm{E}_r^{p,q}$ means page r entry p, q.

We also have mysterious differentials

$$d\colon \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$$

and $\mathrm{E}_{r+1}^{p,q}$ is the cohomology of $\mathrm{E}_r^{\bullet,\bullet}$ at (p, q).

$$\mathrm{E}_{r+1}^{p,q} = \mathrm{H}^{p,q}(\mathrm{E}_r^{\bullet,\bullet}) = \frac{\ker d}{\mathrm{im} d}.$$

We will only deal with sequences where for each (p, q) there is a page r after which all differentials are zero, hence $\mathrm{E}_r^{p,q} = \mathrm{E}_{r+1}^{p,q} = \cdots$. We call this $\mathrm{E}_\infty^{p,q}$.

How do we use all this? In symbols we write

$$\mathrm{E}_2^{p,q} \Rightarrow \mathrm{A}^{p+q}$$

(all our sequences will start at page 2, but that's not always the case). The theory then tells us that the A^n have a magickal complete filtration,

$$0 \subset \cdots \subset \mathrm{F}^{p+2}\mathrm{A}^n \subset \mathrm{F}^{p+1}\mathrm{A}^n \subset \mathrm{F}^p\mathrm{A}^n \subset \cdots \subset \mathrm{A}^n$$

(which in our case will be bounded) whose associated graded pieces are given by the anti-diagonals of $\mathrm{E}_\infty^{p,q}$. Said again in the same way, $\mathrm{E}_\infty^{p,q}$ (for $p + q = n$) is an associated-graded piece of this hidden filtration.

$$\mathrm{E}_\infty^{p+q} = \mathrm{F}^p\mathrm{A}^{p+q}/\mathrm{F}^{p+1}\mathrm{A}^{p+q}.$$

The beauty of it all, is that for the applications we have in mind it doesn't super matter where the differentials and the filtrations come from. We just need to use them to find out who the murderer is.

1.2. **Cohomology of projective space**. A baby example. Consider the fibre bundle[2]

$$\mathbf{C}^\times \hookrightarrow \mathbf{C}^\infty \smallsetminus \{0\} \to \mathbf{CP}^\infty.$$

Since $\mathbf{C}^\infty \smallsetminus \{0\}$ is contractible, Serre applies and we have a spectral sequence

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathbf{CP}^\infty, \mathrm{H}^q(\mathbf{C}^\times)) \Rightarrow \mathrm{H}^{p+q}(\mathbf{C}^\infty \smallsetminus \{0\}).$$

Now we are going to leverage this to extract the cohomology of $\mathbf{CP}^\infty$. This is the opposite of what we were describing in the earlier section. In other words, the author told us who did it and we are trying to figure out how the novel begins.

Here is the E_2 page.

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[2] Recall that $\mathbf{C}^\infty$ is defined as the colimit along inclusions $\mathbf{C}^\mathrm{N} \to \mathbf{C}^{\mathrm{N}+1}$ taking a vector v to $(v, 0)$. Concretely, it consists of sequences $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ of complex numbers, equipped with the weak topology. Abstractly it's defined as the only space X, equipped with maps $\mathbf{C}^k \to \mathrm{X}$, for which giving a map $\mathrm{X} \to \mathrm{Y}$ is equivalent to giving a compatible family of maps $\mathbf{C}^k \to \mathrm{Y}$.

You might think that the product $\mathbf{C}^\mathbf{N}$ might work for this. However, consider the family of maps $\mathbf{C}^k \to \mathbf{C}$ given by summing up the coordinates. This can induce a map on the colimit only if most entries are zeros. Analogously, we define the colimit $\mathbf{CP}^\infty$.

For each k, we have a $\mathbf{C}^\times$-bundle $\mathbf{C}^k \smallsetminus \{0\} \to \mathbf{P}^k$. Together they induce a bundle on $\mathbf{CP}^\infty$. Finally, take the standard representation $\mathbf{C}^\times = \mathrm{GL}_1$. The associated bundle on $\mathbf{P}^k$ is $\mathcal{O}(-1)$.

$$\begin{array}{c|cccc}
1 & \mathrm{H}^0(\mathbf{CP}^\infty) & \mathrm{H}^1(\mathbf{CP}^\infty) & \mathrm{H}^2(\mathbf{CP}^\infty) & \cdots \\
0 & \mathrm{H}^0(\mathbf{CP}^\infty) & \mathrm{H}^1(\mathbf{CP}^\infty) & \mathrm{H}^2(\mathbf{CP}^\infty) & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}$$

The goal is to analyze the $p + q = n$ anti-diagonals.

$$\begin{array}{c|cccc}
1 & \mathrm{H}^0(\mathbf{CP}^\infty) & \mathrm{H}^1(\mathbf{CP}^\infty) & \mathrm{H}^2(\mathbf{CP}^\infty) & \cdots \\
0 & \mathrm{H}^0(\mathbf{CP}^\infty) & \mathrm{H}^1(\mathbf{CP}^\infty) & \mathrm{H}^2(\mathbf{CP}^\infty) & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}$$

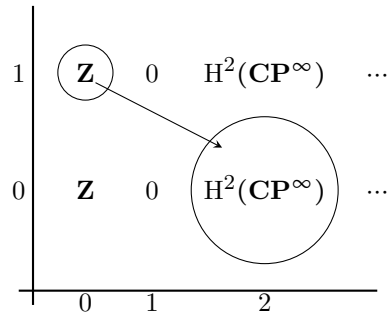
Start with the $p + q = 0$ anti-diagonal. This contains only one guy. As no differentials can come or go from $\mathrm{E}_2^{0,0}$ we have $\mathrm{E}_2^{0,0} = \mathrm{E}_\infty^{0,0}$ and thus $\mathrm{H}^0(\mathbf{CP}^\infty) = \mathrm{H}^0(\mathbf{C}^\infty \setminus \{0\}) = \mathbf{Z}$.

Let's look at $p + q = 1$. There's only one non-zero differential.

$$\begin{array}{c|cccc}
1 & \mathbf{Z} & \mathrm{H}^1(\mathbf{CP}^\infty) & \mathrm{H}^2(\mathbf{CP}^\infty) & \cdots \\
0 & \mathbf{Z} & \mathrm{H}^1(\mathbf{CP}^\infty) & \mathrm{H}^2(\mathbf{CP}^\infty) & \cdots \\
\hline
 & 0 & 1 & 2 &
\end{array}$$

Now, $\mathrm{H}^1(\mathbf{C}^\infty \setminus \{0\}) = 0$ tells us two awesome facts: $\mathrm{H}^1(\mathbf{CP}^\infty) = 0$ and the differential from the picture is injective!

Let's move to $p + q = 2$.



Once again there's one differential to worry about, only this time it's incoming rather than outgoing. Thus, $\mathrm{H}^2(\mathbf{C}^\infty \smallsetminus \{0\}) = 0$ buys us that the differential from the picture must be surjective! Therefore $\mathrm{H}^2(\mathbf{CP}^\infty) = \mathbf{Z}$.

At this point it's clear how to argue by induction, therefore

$$\mathrm{H}^k(\mathbf{CP}^\infty) = \begin{cases} \mathbf{Z} \text{ if } k \text{ even;} \\ 0 \text{ if } k \text{ odd.} \end{cases}$$

1.3. **Cohomology *ring* of projective space.** We want the full cohomology *ring* structure. Here we have to use that the Serre spectral sequence is ***multiplicative.*** In a nutshell, the cup product on cohomology induces a multiplication

$$\mathrm{E}_r^{p,q} \otimes \mathrm{E}_r^{s,t} \to \mathrm{E}_r^{p+s,q+t}$$

on each page. The differentials are ***derivations***:

$$d(xy) = d(x)y + (-1)^{p+q} x d(y)$$

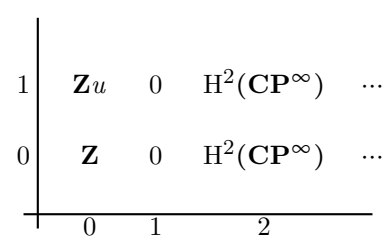
if $x \in \mathrm{E}_r^{p,q}$. The multiplication on $\mathrm{E}_{r+1} = \mathrm{H}(\mathrm{E}_r)$ is the one induced from multiplication on E_r. Finally, if the spectral sequence converges to A^n, the latter has a multiplication which restricts to each filtered piece, inducing a multiplication on the associated gradeds, which in turn coincides with the one on E_∞.

In our special case we also have graded commutativity:

$$xy = (-1)^{(p+q)(s+t)} yx$$

for $x \in \mathrm{E}_r^{p,q}$, $y \in \mathrm{E}_r^{s,t}$.

Let's see how to use this for $\mathbf{CP}^\infty$. Go back to the E_2 page.



Denote by u a generator of $\mathrm{H}^1(\mathbf{C}^\times)$. We know the differential (from the previous picture) is an isomorphism, so $x := du$ is a generator of $\mathrm{H}^2(\mathbf{CP}^\infty)$.

Lemma 1.4 – The element ux is a generator of $\mathrm{H}^2(\mathbf{CP}^\infty)$. *

This should follow by a sort of "twisted Kunneth" theorem. Basically, the second page is just $\mathrm{E}_2 = \mathrm{H}^*(\mathbf{CP}^\infty) \otimes \mathrm{H}^*(\mathbf{C}^\times)$. Let's treat this as a black box.

Repeating the argument above, I know $d(ux)$ is a generator for $\mathrm{H}^4(\mathbf{CP}^\infty)$. By Leibniz

$$d(ux) = d(u)x + ud(x) = x^2.$$

So the cup product of x with itself generates $\mathrm{H}^4(\mathbf{CP}^\infty)$! In general, I repeat by observing that ux^n is a generator of $\mathrm{H}^{2n}(\mathbf{CP}^\infty) \otimes \mathrm{H}^1(\mathbf{C}^\times)$ and $d(ux^n) = x^{n+1}$ is a generator of $\mathrm{H}^{2n+2}(\mathbf{CP}^\infty)$.

Thus,

$$\mathrm{H}^*(\mathbf{CP}^\infty) = \mathbf{Z}[x].$$

1.4. **Cohomology of actual projective space.** Oddly enough, computing cohomology of $\mathbf{CP}^\mathrm{N}$ seems slightly more annoying.[3]

We have the fibration

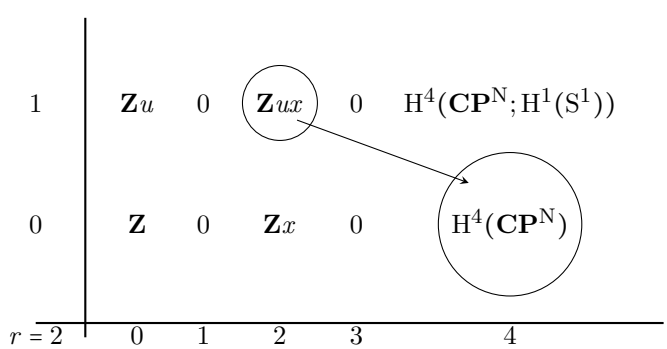
$$\mathbf{C}^\times \to \mathbf{C}^{\mathrm{N}+1} \smallsetminus \{0\} \to \mathbf{CP}^\mathrm{N}$$

which, up to homotopy, is the same as

$$\mathrm{S}^1 \to \mathrm{S}^{2\mathrm{N}+1} \to \mathbf{CP}^\mathrm{N}$$

(the case $\mathrm{N} = 1$ is so important it has its own name, it's called the *Hopf fibration*). We are going to cheat and use, say, a cellular decomposition of $\mathbf{CP}^\mathrm{N}$ to know its cohomology groups. We are going to use the SSS to deduce the ring structure.

Let's have a look at the E_2 page. As before, using Kunneth (and freeness of $\mathrm{H}^*(\mathrm{S}^1)$) we have that as a bigraded ring $\mathrm{E}_2 = \mathrm{H}^*(\mathbf{CP}^\mathrm{N}) \otimes \mathrm{H}^*(\mathrm{S}^1)$. We use notation $\mathrm{H}^*(\mathrm{S}^1) = \mathbf{Z}[u]/(u^2)$, with $\deg u = 1$. Once again, since $\mathrm{H}^i(\mathrm{S}^{2\mathrm{N}+1}) = 0$, for $i = 1, 2$, we have that $x := d(u)$ is a generator of $\mathrm{H}^2(\mathbf{CP}^\mathrm{N})$.



If $\mathrm{N} = 1$ we are done, $\mathrm{H}^*(\mathbf{CP}^1) = \mathbf{Z}[x]/(x^2)$ with $\deg x = 2$.

If $\mathrm{N} = 2$, then $\mathrm{H}^4(\mathrm{S}^5) = 0$ means $d(ux)$ is a generator of $\mathrm{H}^4(\mathbf{CP}^2)$. But by Leibniz $d(ux) = x^2$. So $\mathrm{H}^*(\mathbf{CP}^2) = \mathbf{Z}[x]/(x^3)$.

For $\mathrm{N} = 3$ we do the same trick. First we observe that ux^2 is a generator for $\mathrm{H}^4(\mathbf{CP}^\mathrm{N}; \mathrm{H}^1(\mathrm{S}^1))$. Then use vanishing of $\mathrm{H}^6(\mathrm{S}^7)$ and Leibniz to conclude that $d(ux^2) = x^3$ generates $\mathrm{H}^6(\mathbf{CP}^3)$.

[3] In the sense that the only way I know how to do it is to know already that $\mathrm{H}^k(\mathbf{CP}^\mathrm{N}) = 0$ for $k > 2\mathrm{N}$. However, once you know $\mathbf{CP}^\infty$, I believe you can bootstrap the cohomology of $\mathbf{CP}^\mathrm{N}$ by using the fibration $\mathrm{S}^{2\mathrm{N}+1} \to \mathbf{CP}^\mathrm{N} \to \mathbf{CP}^\infty$. I haven't checked the details.

At this point it's clear the same trick works for any N. So, $\mathrm{H}^*(\mathbf{CP}^{\mathrm{N}}) = \mathbf{Z}[x]/(x^{\mathrm{N}+1})$ with $\deg x = 2$.

1.5. **Finally the homotopy group**. First, we need a little bit of theory. Every space X has an associated *Whitehead tower*

$$\mathrm{X} = \mathrm{W}_0 \leftarrow \mathrm{W}_1 \leftarrow \mathrm{W}_2 \leftarrow \mathrm{W}_3 \leftarrow \cdots$$

where

- $\pi_k(\mathrm{W}_n) = 0$, for $k \leq n$.
- $\pi_k(\mathrm{W}_n) = \pi_k(\mathrm{X})$ for $k > n$.
- $\mathrm{W}_k \to \mathrm{W}_{k-1}$ is a fibration with fibre $\mathrm{K}(\pi_k\mathrm{X}, k-1)$.

Recall that for an abelian group G there exists a (unique up to homotopy) space $\mathrm{K}(\mathrm{G}, n)$ with the property that

$$\pi_k\mathrm{K}(\mathrm{G}, n) = \begin{cases} \mathrm{G} & k = n \\ 0 & k \neq n. \end{cases}$$

Basically we are killing off one by one the homotopy groups of X from below.

Remark 1.5. This is the analogue of the *truncation* construction for chain complexes (which one uses often when dealing with derived categories). Namely, given a chain complex

$$\cdots \mathrm{E}_k \to \mathrm{E}_{k-1} \to \mathrm{E}_{k-2} \to \cdots$$

we can form its truncation $\tau_{<n}(\mathrm{E})$ by defining

$$\cdots \to \mathrm{E}_{n+3} \to \mathrm{E}_{n+2} \to \ker(\mathrm{E}_{n+1} \to \mathrm{E}_n) \to 0 \to \cdots$$

which has the property that

$$\mathrm{H}_k(\mathrm{E}) = \begin{cases} 0 & k \leq n \\ \mathrm{H}_k(\mathrm{E}) & k > n. \end{cases}$$

We have a map $\tau_{<n+1}\mathrm{E} \to \tau_{<n}\mathrm{E}$ whose cokernel is just $\mathrm{H}_n(\mathrm{E})[n]$ (and if you apply Dold-Kan to it you obtain precisely $\mathrm{K}(\mathrm{H}_n(\mathrm{E}), n-1)$.

In chain complexes it's plain to see that there are also opposite truncations $\tau_{>n}$, using coker. In topology this corresponds to killing off the higher homotopy groups. The corresponding gadget is called the *Postnikov tower.* *

OK, so let's have a look at $\mathrm{X} = \mathrm{S}^3$. Since $\pi_1 = \pi_2 = 0$ the tower really only starts at 3. So we have a fibration

I may or may not have gotten the indices wrong in this last one.

$$\mathrm{K}(\pi_3\mathrm{S}^3, 2) \to \mathrm{W} \to \mathrm{S}^3$$

where $\pi_1\mathrm{W} = \pi_2\mathrm{W} = \pi_3\mathrm{W} = 0$. Recall that $\pi_3\mathrm{S}^3 = \mathbf{Z}$, so we need to get a handle on $\mathrm{K}(\mathbf{Z}, 2)$. But we know who who this guy is, $\mathrm{K}(\mathbf{Z}, 2) = \mathbf{CP}^\infty$!

This follows from the long exact sequence in homotopy for a fibration.

Theorem 1.6 – Let $\mathrm{F} \to \mathrm{E} \to \mathrm{B}$ be a fibration. Then we have a long exact sequence in homotopy

$$\cdots \to \pi_2\mathrm{B} \to \pi_1\mathrm{F} \to \pi_1\mathrm{E} \to \pi_1\mathrm{B} \to \pi_0\mathrm{F} \to \pi_0\mathrm{E} \to \pi_0\mathrm{B} \to 0$$

where the last few terms are just pointed sets. *

Now, consider the fibration $S^1 \to S^\infty \to \mathbf{CP}^\infty$. We have the long exact sequence

$$0 \to \pi_2 \mathbf{CP}^\infty \to \mathbf{Z} \to 0 \to \pi_1 \mathbf{CP}^\infty \to 0$$

where we used that S^∞ is contractible. Hence $\pi_k \mathbf{CP}^\infty$ is zero for $k \neq 2$ and $\mathbf{Z}$ for $k = 2$. In other words, $\mathbf{CP}^\infty = K(\mathbf{Z}, 2)$.

Theorem 1.7 (Hurewicz) – Let X be path connected and such that $\pi_k X = 0$ for $0 < k < n$. Then $H_k(X) = 0$ for $0 < k < n$ and the natural map $\pi_n(X) \to H_n X$ is an isomorphism (unless $n = 1$, in which case it's the abelianization quotient map). Moreover, $\pi_{n+1} X \to H_{n+1}(X)$ is surjective.

As corollary, we may say that if X is simply connected and $H_i(X) = 0$ for $0 < i < n$, then $\pi_i(X) = 0$ for $i < n$. *

Theorem 1.8 (Universal Coefficients) – For X, we have a short exact sequence

$$0 \to \mathrm{Ext}^1(H_{k-1} X, \mathbf{Z}) \to H^k(X) \to \mathrm{Hom}(H_k(X), \mathbf{Z}) \to 0.$$

This becomes even more useful when we know the homology groups are finitely generated. Indeed, any abelian group A fits in a short exact sequence

$$0 \to T_A \to A \to F_A \to 0$$

where T_A is torsion and F_A free (hence the sequence splits, non-canonically). In particular, if we write $H_i(X) = \mathbf{Z}^{b_i} \oplus T_i$ where T_i is torsion, we have

$$H^i(X) = \mathbf{Z}^{b_i} \oplus T_{i-1}$$

(non-canonically of course). *

[4]

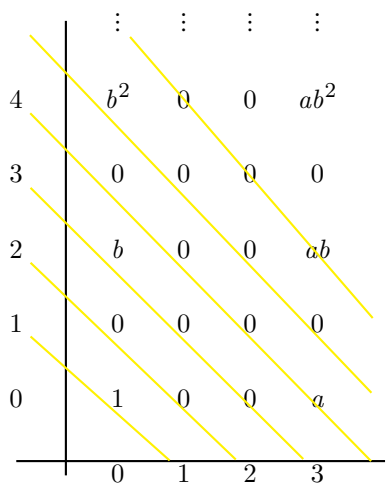
OK, let's see what the Serre spectral sequence will tell us. First off, Hurewicz says $H_k(W) = 0$ for $0 < k \leq 3$ and therefore $H^k(W) = 0$ for the same k, by universal coefficients.

Let's go back to the SSS.

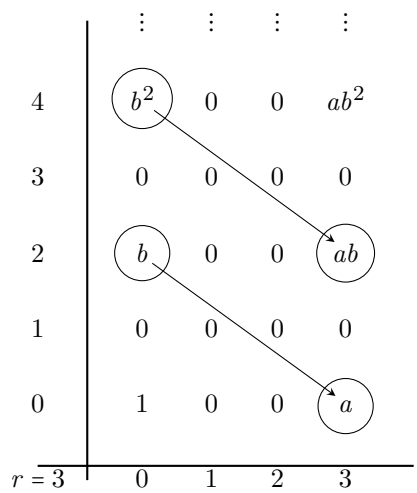
$$E_2^{p,q} = H^p(S^3; H^q(\mathbf{CP}^\infty) \Rightarrow H^{p+q}(W)$$

As before, we use the freenes of the cohomology of the fibre plus Kunneth to imply that the bigraded ring E_2 is actually isomorphic to $H^*(S^3) \otimes H^*(\mathbf{CP}^\infty)$. Call a the generator of $H^3(S^3)$ and b the generator of $H^2(\mathbf{CP}^\infty)$. Let's get down to drawing the spectral sequence (I'll just write down the generators, omitting the $\mathbf{Z}$). I'll highlight the diagonals first.

[4] The fact that $\mathrm{Ext}^1(T, \mathbf{Z}) = T$ I think can be deduced from the structure theorem. Since T is finitely generated torsion abelian group, it's a product of cyclic groups. A cyclic group has a resolution $\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. Take the product of all these, you get a resolution $\mathbf{Z}^r \to \mathbf{Z}^r \to T$ where the first map is componentwise multiplication by the order of the corresponding cyclic group. Applying $\mathrm{Ext}^1(-, \mathbf{Z})$ we find that $\mathrm{Ext}^1(T, \mathbf{Z})$ is again the quotient of the same map $\mathbf{Z}^r \to \mathbf{Z}^r$. Here I've used that the transpose of the multiplication by n map $\mathbf{Z} \to \mathbf{Z}$ is again multiplication by n.



The only page where the differentials can be non-zero is the third, let's draw those.



Since $\mathrm{H}^2(\mathrm{W}) = 0 = \mathrm{H}^3(\mathrm{W})$ we have that the first differential is zero and thus (up to switching a sign) we can assume $a = d(b)$.

Now (FINALLY!) we'll use the spectral sequence to deduce something about the total space! The Leibniz rule (plus graded commutativity) tells us that

$$d(b^2) = d(b)b + bd(b) = 2ab.$$

Hence the differential $d\colon \mathrm{E}_3^{4,0} \to \mathrm{E}_3^{2,3}$ is injective. It follows that $\mathrm{H}^4(\mathrm{W}) = 0$. But we also know what the cokernel of d is (since the map is just multiplication by 2), so we have $\mathrm{H}^5(\mathrm{W}) = \mathrm{coker} = \mathbf{Z}/2\mathbf{Z}$. Nice!

We can also compute the rest of the cohomology of W by looking at the differentials. It's zero in odd degrees and $\mathbf{Z}/n\mathbf{Z}$ in degree $2n$ (I think). But we won't need all this extra information.

OK, how is this relevant? Well, by Hurewicz $\pi_4 W = H_4(W)$. Since $H^4(W) = 0$ we have that $H_4(W)$ must be torsion and hence it must be equal to the torsion of $H^5(W)$. But we computed this, so

$$\pi_4 S^3 = \pi_4 W = H_4(W) = \mathbf{Z}/2\mathbf{Z}.$$

Using the Hopf fibration (and the associated long exact sequence in homotopy), we can moreover deduce that $\pi_4 S^3 = \pi_4 S^2$. In particular,

$$\pi_4 S^2 = \mathbf{Z}/2\mathbf{Z}.$$