Sheaves, covering spaces, monodromy and an application

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Abstract. All mistakes here are mine, you cannot have them. Check out Szamuely's book on covering spaces and Galois theory (and also Arapura's book on algebraic geometry).

1. Monodromy

Sheaves are widely used in algebraic geometry. However, the ones we typically encounter are the *coherent* ones. Nevertheless, the sheaves from topology are also super-useful (but of very different nature). Let me say a word or two about those.

1.1. **Coverings and sheaves**. To any map of spaces $\pi\colon E \to X$ there is an associated *sheaf of sections* $\mathscr{E}$ defined by (I'm assuming all maps are continuous here)

$$\mathscr{E}(U) = \{s\colon X \to E \mid \pi s = \mathrm{id}_X\}$$

where U ranges over all the open sets of X and the restriction maps are given by restriction of functions.

Example 1.1 – The simplest example of this is when $E = X \times F$ where F is a discrete set. In this case I'll write F_X for the corresponding sheaf, which assigns to U the set of locally constant functions $s\colon U \to F$. In particular, if U is connected, $F_X(U) = F$. Hence, F_X is called the *constant sheaf* with fibre F.[1]   ∗

Example 1.2 – Here's a more interesting example. Let $E \to X$ be a covering space. Non-canonically, for each $x \in X$, all fibres E_x are isomorphic to a fixed discrete set F. Moreover, there exists an open cover of X, such that on each patch U, $E|U$ becomes $U \times F$. This structure can be seen at the level of the corresponding sheaf $\mathscr{E}$. Indeed, the same trivializing open cover tells us that on each open patch U, the restriction $\mathscr{E}|U$ is isomorphic to the constant sheaf F_U. Sheaves satisfying this property are called *locally constant*.   ∗

We can also go the other way around. Given a locally constant sheaf $\mathscr{E}$ with fibre F, we can produce a covering space as follows. Let $\{U_i\}_i$ be an open cover with isomorphisms $\alpha_i\colon \mathscr{E}|U_i \simeq F_{U_i}$. On the double overlaps $U_{ij} = U_i \cap U_j$ we then have isomorphisms $\alpha_{ij}\colon F_{U_{ij}} \to F_{U_{ij}}$ obtained by composing (the restrictions of) α_i^{-1} and α_j. But what is a morphism of sheaves? Unpacking everything we see that we are left with a bunch of maps $g_{ij}\colon U_{ij} \to \mathrm{Aut}(F)$ satisfying the cocycle condition. In other words, we have given the cocycle presentation for a covering space (both gadgets are parameterized by $\check{H}^1(X, \mathrm{Aut}(F))$). Actually, there is an equivalence of categories

$$\mathrm{Cov} = \mathrm{LCS}$$

between covering spaces of X and locally constant sheaves (of sets). Great!

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[1] I guess I should probably say *stalk* rather than fibre.

1.2. **Monodromy**. Recall this elementary theorem from topology.

Theorem 1.3 (Monodromy theorem) – Let $f: E \to X$ be a covering space and let $\gamma: I \to X$ be a loop. Let $y \in E$ be such that $f(y) = \gamma(0)$. Then there exists a unique lift $\tilde{\gamma}: I \to E$. In other words, $f\tilde{\gamma}(t) = \gamma(t)$. Moreover, if $\gamma \sim \gamma'$ (ie they are homotopic) then $\tilde{\gamma} \sim \tilde{\gamma'}$ (rel the endpoints). Note that $\tilde{\gamma}$ implicitly depends on the choice of y upstairs.   ∗

Fix x_0, the above can be souped up by saying there is an action $\pi_1 = \pi_1(X, x_0)$ on the fibre $F = E_{x_0}$. This action is given by

$$\gamma \cdot y = \tilde{\gamma}(1)$$

which is called the monodromy action.

We can also go back. Let F be a π_1-set[2] then we use the universal cover $\tilde{X} \to X$ to produce the covering space

$$\tilde{X} \times F / \pi_1 \to X$$

where π_1 is acting diagonally (it acts on $\tilde{X}$ by deck transformations, which is the same as monodromy). As before, this determines an equivalence $\texttt{Cov} = \pi_1\texttt{-Set}$ between covering spaces and π_1-sets. If you want to get a hold directly of the associated sheaf, it's given by

$$\mathscr{F}(U) = \{ g: \tilde{U} \to F \mid g \text{ equivariant } \}$$

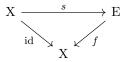where $\tilde{U}$ is the preimage of U under $\tilde{X} \to X$.[3] Anyway, we have

$$\texttt{LCS} = \texttt{Cov} = \pi_1\texttt{-Set}.$$

1.3. **Global sections**. The nice thing about this equivalence is that we can translate what global sections of a sheaf are (which is the baby version of cohomology). So, given a sheaf $\mathscr{G}$ the global sections are denoted by

$$\mathscr{G}(X) = \Gamma(X, \mathscr{G}) = H^0(X, \mathscr{G}).$$

Take $f: E \to X$ to be the corresponding covering space. Then an element $s \in \mathscr{G}(X)$ is a map $s: X \to E$ such that fs is the identity. Diagrammatically we write the following.

$$X \xrightarrow{\quad s \quad} E$$
$$\text{id} \searrow \qquad \swarrow f$$
$$X$$

Taking monodromy with respect to a fixed base point x_0, f corresponds to a π_1-set F. We see that a global section s corresponds to a(n equivariant) map

$$\bullet \to F$$

from the trivial π_1-set to F. In other words,

$$\mathscr{G}(X) = F^{\pi_1}$$

global sections are fibre elements invariant under monodromy!

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[2] Which I think we must assume to be non-empty, unless we are allowing the empty set as a covering space. Have a look at this $http://www.renyi.hu/\ szamuely/\text{CLS}z.pdf$.

[3] Unfortunately I don't know of a more direct way to do this. In particular, for a *finite* π_1-set what I wrote above is true for the étale topology in algebraic geometry (and so without using universal covers).

Remark 1.4. With some category theory cheat, the discussion above can be cut short. Consider the constant sheaf ϵ given by assigning to each U a singleton. This is clearly the *final object* of the category Sh of sheaves. It's plain to see that $\mathscr{G}(X) = \mathrm{Hom}(\epsilon, \mathscr{G})$. Hence, global sections are the same as maps from the final object.

Since Sh = Cov = π_1-Set (and being a final object is a categorical notion) we are reduced to spotting who the final object is in each category, which is obvious.   *

1.4. **Cohomology**. Let's now consider sheaves of abelian groups. For example, take the constant sheaf $\mathbf{Z}_X$ on a space X. Or take whatever other sheaf $\mathscr{G}$ of sets and apply the free abelian group functor.

Since we've thrown in abelian groups, we've gain the ability to perform addition (and subtraction). Hence we can define cohomology, which is a very useful gadget. Formally, $\mathrm{H}^p(X; \mathscr{G})$ is the p-th derived functor of the global sections functor (which as we say earlier can also be described as $\mathrm{Hom}(\bullet, \mathscr{G})$, where $\bullet$ is the trivial sheaf, so in some sense $\mathrm{H}^p(X; \mathscr{G}) = \mathrm{Ext}^i(\bullet, \mathscr{G})$.

The theory goes through a lot of abstract stuff (picking injective resolutions and whatnot) to define these. In practice one uses exact sequences to reduce to known cases, which are proved by combinatorial methods (for example simplicial or Cech cohomology). Notice that once again $\mathrm{H}^0(X, \mathscr{G}) = \mathscr{G}(X) = \Gamma(X, \mathscr{G})$.

What we care about is that in the special case of F being the constant sheaf $\mathbf{Z}_X$, then $\mathrm{H}^i(X; \mathbf{Z}_X)$ is precisely singular cohomology with $\mathbf{Z}$ coefficients. This is explained (for example) in Warner's book on differentiable manifolds. However, we'll see below that cohomology with so-called *twisted coefficients* is also useful.

1.5. **Pushforward**. Given a map of spaces $f: X \to Y$ we can "push forward" sheaves. We define $f_* \mathscr{G}$ by the formula

$$f_* \mathscr{G}(U) = \mathscr{G}(f^{-1}(U))$$

and so get a functor $\mathrm{Sh}_X \to \mathrm{Sh}_Y$. The cool thing is that f_* also has right derived functors $\mathrm{R}^p f_*$. Concretely, we can define $\mathrm{R}^p f_* \mathscr{G}$ as the sheaf associated with the assignment

$$U \mapsto \mathrm{H}^p(f^{-1}U, \mathscr{G}|f^{-1}U)$$

where $\mathscr{G}|f^{-1}U$ means considering $\mathscr{G}$ as a sheaf on the open $f^{-1}U$. We will see in a second why $\mathrm{R}^p f_*$ should be thought of as "taking cohomology of the fibres".

1.6. **Local systems**. Fix an abelian group G. For any X we have a constant sheaf G_X. A *local system* (with fibre G) is a sheaf of abelian groups which locally is isomorphic to the constant sheaf G. We can upgrade the equivalence LCS = π_1-Set to

$$\mathrm{Loc} = \mathrm{Rep}\,\pi_1$$

where the LHS is the category of local systems (we are not fixing a G) and the RHS is the category of representations of π_1 in abelian groups.[4]

Given $f: E \to X$ and a local system $\mathscr{G}$, the pushforward $f_* \mathscr{G}$ is not necessarily a local system. But sometimes it is!

Example 1.5 – Let $f: E \to X$ be a fibre bundle with fibre F and consider $\mathscr{G}$ to be $\mathrm{R}^p f_* \mathbf{Z}$ for some p. Assume X to be a manifold, so it has a basis made up of balls (and we can assume they trivialize f). Thus, $\mathrm{R}^p f_* \mathbf{Z}(U) = \mathrm{H}^p(U \times F, \mathbf{Z}) = \mathrm{H}^p(F, \mathbf{Z})$ is fixed under shrinking U (I guess one should make sure the identifications above are indeed canonical). So, $\mathrm{R}^p f_* \mathbf{Z}$ is a local system!   *

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[4] Notice here the issue with the empty π_1-set is non-existent.

Since these guys above are local systems, we have corresponding π_1-representation. So,

$$\mathrm{H}^0(\mathrm{X}, \mathrm{R}^p f_* \mathbf{Z}) = \mathrm{H}^p(\mathrm{F}, \mathbf{Z})^{\pi_1(\mathrm{X})}.$$

1.7. **Elliptic curves**. Let's see a quick application of the concepts above to algebraic geometry. An elliptic curve E is always a quotient $\mathbf{C}/\Lambda$. More intrinsically, we have an exact sequence

$$0 \to \mathrm{H}_1(\mathrm{E}, \mathbf{Z}) \to \mathrm{H}^0(\mathrm{E}, \Omega)^\vee \to \mathrm{E} \to 0$$

where

$$\gamma \mapsto \left[\omega \mapsto \int_\gamma \omega \right].$$

Sheaves are handy in families. Suppose you have $f \colon \mathrm{E} \to \mathrm{X}$ where E, X are complex manifolds and the fibres of f are elliptic curves. Then the family version of the sequence above is

$$0 \to \underline{\mathrm{Hom}}_{\mathbf{Z}_{\mathrm{X}}}(\mathrm{R}^1 f_* \mathbf{Z}_{\mathrm{E}}, \mathbf{Z}_{\mathrm{X}}) \to \underline{\mathrm{Hom}}_{\mathcal{O}_{\mathrm{X}}}(f_* \Omega_{\mathrm{E}/\mathrm{X}}, \mathcal{O}_{\mathrm{X}}) \to \mathrm{E}$$

so we can describe a family of elliptic curves in terms of an inclusion from a local system to a vector bundle. I think this means that families of elliptic curves are described by a flat vector bundles (or by a D-module).

1.8. **Leray spectral sequence**. Again, take a sheaf $\mathcal{G}$ and a map $f \colon \mathrm{E} \to \mathrm{X}$. I want to relate the cohomology $\mathcal{G}$ with the cohomology of $f_* \mathcal{G}$. To relate the two gadgets there is a spectral sequence. The correct way to phrase this would be to use derived categories. One writes

$$\mathrm{H}^*(\mathrm{E}, \mathcal{G}) = \mathrm{H}^*(\mathrm{X}, \mathrm{R}f_* \mathcal{G})$$

which is an isomorphism of chain complexes of abelian groups. Here $\mathrm{R}f_*$ is the *total derived functor* of f_*, which is a chain complex of sheaves of abelian groups on X. Its cohomology sheaves are given by $\mathrm{H}^p(\mathrm{R}f_* \mathcal{G}) = \mathrm{R}^p f_* \mathcal{G}$.

In practice what we need in computations is what is called the *Leray spectral sequence* which in symbols is

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p(\mathrm{X}, \mathrm{R}^q f_* \mathcal{G}) \Rightarrow \mathrm{H}^{p+q}(\mathrm{E}, \mathcal{G}).$$

1.9. **Covering spaces again**. Let's have a look at a special case. Take a covering space $\mathrm{E} \to \mathrm{X}$ with fibre the discrete space F. Take the constant sheaf $\mathbf{Z}_{\mathrm{E}}$. Notice that since F is discrete it has no higher cohomology, hence $\mathrm{R}^i f_* \mathbf{Z} = 0$ for $i > 0$. Hence the spectral sequence collapses and just says that

$$\mathrm{H}^i(\mathrm{E}) \simeq \mathrm{H}^i(\mathrm{X}, f_* \mathbf{Z}).$$

I find this really nice, it's really showing how the local system $f_* \mathbf{Z}$ (which is a "twisting" of the constant sheaf $\mathbf{Z}$) picks up on all the cohomology of E.

2. A computation of Betti numbers

I got this from Arapura's amazing introductory book on algebraic geometry. Let $f \colon \mathrm{E} \to \mathrm{C}$ be a map between smooth projective varieties. Assume C to be a curve, E a surface, and the fibres of f to be connected. Let $\Delta \subset \mathrm{C}$ be the locus where the fibres of f are singular and let $\mathrm{U} = \mathrm{C} \smallsetminus \Delta \subset \mathrm{C}$ be the complent. Since $f|f^{-1}\mathrm{U}$ is a submersion between smooth manifolds it must be a fibre bundle (this is Ehresmann's theorem and it's about manifolds, so there's no claim about the complex structure). In any case, the topological fibre over U is some fixed topological surface F. Then

$$b_1(\mathrm{C}) \le b_1(\mathrm{E}) \le b_1(\mathrm{C}) + \dim \mathrm{H}^1(\mathrm{F}, \mathbf{Q})^{\pi_1(\mathrm{U})}.$$

Let's prove this. The first inequality follows from Hodge theory. Indeed, pullback cannot kill forms, so $b_1(C) = 2h^{1,0}(C) \leq 2h^{1,0}(E) = 2b_1(E)$.

For the second inequality we start by observing that a very crude estimate coming out of the Leray spectral sequence is that

$$\dim H^1(C, \mathbf{Q}) \leq \dim H^1(C, f_*\mathbf{Q}) + \dim H^0(C, R^1 f_*\mathbf{Q}) = b_1(C) + \dim H^0(C, R^1 f_*\mathbf{Q}).$$

Let's study the RHS. Let's draw a silly diagram.

$$
\begin{array}{ccc}
f^{-1}U & \longrightarrow & E \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
U & \xrightarrow{\ j\ } & C
\end{array}
$$

Let $\mathscr{G} = R^1 f_* \mathbf{Q}$. Let $j\colon U \to C$ be the inclusion. We call $j^*\mathscr{G}$ the restriction of $\mathscr{G}$ to open subsets of U. By chasing definitions it follows that

$$j_* j^* \mathscr{G} = R^1 g_* \mathbf{Q}.$$

By Ehresmann (and discussions above) we know that $j^*\mathscr{G}$ is a local system, with fibre $H^1(F, \mathbf{Q})$. Hence

$$H^0(C, j_* j^* \mathscr{G}) = H^1(F, \mathbf{Q})^{\pi_1(U)}.$$

Claim 2.1 – There is a natural map $\mathscr{G} \to j_* j^* \mathscr{G}$. This map is an injection. ∗

This follows from staring at the definitions of the two sheaves and a general lemma (which we won't prove).

Lemma 2.2 – Let $W \subset X$ be a codimension one subvariety of a variety X. Then $H^1(X, \mathbf{Q}) \to H^1(W, \mathbf{Q})$ is injective. (Arapura argues that when you can cover W by a finite number of balls you can use Mayer-Vietoris). ∗

Using the claim we have that $H^0(C, \mathscr{G}) \to H^0(C, j_* j^* \mathscr{G})$ must be injective, thus

$$b_1(E) \leq b_1(C) + \dim H^0(C, \mathscr{G}) \leq b_1(C) + \dim H^0(C, j_* j^* \mathscr{G}) = b_1(C) + \dim H^1(F, \mathbf{Q})^{\pi_1(U)}.$$

Example 2.3 – Let $E \to C$ be an elliptic surface with non-trivial monodromy. Then $b_1(E) = b_1(C)$. ∗

This can be deduced as follows. The fibre F over U is an elliptic curve, so $H^1(F, \mathbf{Q})$ is 2-dimensional. Since monodromy is non-trivial, we must have $\dim H^1(F, \mathbf{Q}) \leq 1$. Thus, $b_1(E) \leq b_1(C) + 1$. But we also have $b_1(C) \leq b_1(E)$ and, by Hodge theory, $b_1(E)$ must be even, so $b_1(E) = b_1(C)$.