whatsaVermaModule?

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Abstract. In Fall 2017 I taught a course on Lie theory. I wrote notes on Verma modules. These are said notes. All mistakes here are mine and mine only: you cannot have them.

How do you cook up representations of a Lie algebra $\mathfrak{g}$? One way is using Verma modules.

1. Tensor Products

Let V be a vector space. The ***tensor algebra*** is $\mathrm{TV} = \bigoplus_{k\geq 0} \mathrm{V}^{\otimes k}$. Notice that $\mathrm{V}^{\otimes 0} = \mathbf{C}$. TV is an associative (unital) algebra. It's also the "biggest" possible algebra you can cook up from V, in the sense that it is free on $\dim \mathrm{V}$ generators. Concretely, we have a bijection

$$\mathrm{Ass}(\mathrm{TV}, \mathrm{B}) = \mathrm{Vect}(\mathrm{V}, \mathrm{B})$$

where B is any other associative (unital[1]) algebra. So, this means that to specify an algebra map $\mathrm{TV} \to \mathrm{B}$ it suffices to give a linear map $\mathrm{V} \to \mathrm{B}$. In other words, $\mathrm{T}\colon \mathrm{Vect} \to \mathrm{Ass}$ is "left adjoint" to the forgetful functor $\mathrm{Ass} \to \mathrm{Vect}$ which forgets multiplication (i.e. takes an algebra and views it as a plain vector space).

The tensor algebra is the mother of many constructions. It can be used for a lot of fun stuff. For example, what if we need a *commutative* algebra built out of V? Well, we take TV and impose commutativity. Let $\mathrm{I} \subset \mathrm{TV}$ be the (bilateral) ideal generated by elements

$$v \otimes w - w \otimes v$$

for all $v, w \in \mathrm{V}$. The quotient TV/I is precisely the ***symmetric algebra*** SV, which is a commutative algebra. It satisfies the universal property

$$\mathrm{CAlg}(\mathrm{SV}, \mathrm{B}) = \mathrm{Hom}(\mathrm{V}, \mathrm{B})$$

where now B is a commutative algebra. So, $\mathrm{S}\colon \mathrm{Vect} \to \mathrm{CAlg}$ is once again left adjoint to the forgetful functor $\mathrm{CAlg} \to \mathrm{Vect}$.

What if you need a Grasmann algebra? Take J to be the ideal generated by elements

$$v \otimes v$$

for all $v \in \mathrm{V}$. This is the ***exterior algebra*** $\Lambda\mathrm{V}$, which is universal among all algebras B such that $b^2 = 0$ for any $b \in \mathrm{B}$.

If q is a quadratic form on V, we can form the ***Clifford algebra*** $\mathrm{C}l(\mathrm{V})$. This is defined by taking the the ideal K generated by elements

$$v \otimes v - q(v).$$

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[1] I'll stop saying unital now.

Notice that these relations are not homogeneous (with respect to the grading on TV given by word-length), so that $Cl(V)$ is not $\mathbf{Z}$-graded (as opposed to all examples before) but only $\mathbf{Z}/2\mathbf{Z}$-graded (i.e. you only remember the parity of a tensor). Clifford algebras can be used to construct the double cover of $SO(n)$, Spin_n. For this, see for example Sepanski's book.

2. Enveloping algebra

If $V = \mathfrak{g}$ is a Lie algebra, we can feed that into our woodchipper that is TV. Consider the ideal H of TV generated by elements

$$x \otimes y - y \otimes x - [x, y]$$

for all $x, y \in \mathfrak{g}$. The quotient $U(\mathfrak{g}) = TV/H$ is called the ***universal enveloping algebra*** of $\mathfrak{g}$. It is an associative algebra, satisfying the universal property

$$\mathrm{Ass}(U(\mathfrak{g}), B) = \mathrm{Lie}(\mathfrak{g}, B)$$

where B is an associative algebra. In other words, there is a functor $\mathrm{Ass} \to \mathrm{Lie}$ by taking an associative algebra and viewing it as a Lie algebra (using the commutator bracket), and U is the left adjoint. Once again, the relations aren't homogeneous so that $U(\mathfrak{g})$ is not graded. It is *filtered*, however, and it turns out that the associated graded is $S\mathfrak{g}$, the symmetric algebra.

Before we move on, let us fix some notation. There is a natural map : $\mathfrak{g} \to T\mathfrak{g} \to U(\mathfrak{g})$, which turns out is an injection. So we identify elements $x \in \mathfrak{g}$ with elements of $U(\mathfrak{g})$. If $\alpha, \beta \in U(\mathfrak{g})$ we will write $\alpha\beta$ for their product in $U(\mathfrak{g})$, instead of $\alpha \otimes \beta$.

Remark 2.1. A silly but important observation is the following. A $\mathfrak{g}$-representation is nothing but a Lie algebra map $\mathfrak{g} \to \mathrm{End}(V)$. By the universal property, this is the same as a map $U(\mathfrak{g}) \to \mathrm{End}(V)$ of associative algebras. Such a map is the same as a $U(\mathfrak{g})$-module structure on V. Hence

$$\mathtt{Rep}(\mathfrak{g}) = \mathtt{Mod}(U(\mathfrak{g})).$$

So, any module we can cook up for $U(\mathfrak{g})$ will automatically provide us with a representation of $\mathfrak{g}$.

3. Verma modules

What's the goal? Let $\mathfrak{g}$ be semi-simple now. Let us also fix a base Δ for later, so that we may speak of positive and negative roots $R^{\pm}$. Recall that, as a vector space, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$$

For any weight $\mu \in \mathfrak{h}^\vee$, we are trying to find a highest weight cyclic representation π_μ of $\mathfrak{g}$ with highest weight μ. This means there should be a weight vector v_0 such that

- $(\pi_\mu(H))v_0 = \mu(H)v_0$, for all $H \in \mathfrak{h}$ [as we want v_0 to be a weight vector]
- $(\pi_\mu X)v_0 = 0$, for all $X \in \mathfrak{g}_\alpha$ with $\alpha \in R^+$ [as we want v_0 to have highest weight]
- We want the whole vector space to be spanned by iterated applications of the $\pi_\mu(Y)$ operators to v_0, for $Y \in g_\alpha$ with $\alpha \in R^-$ [as we want v_0 to be a cyclic vector]

So one thing we could do is just impose these relations on $\mathrm{U}(\mathfrak{g})$. But let's give a slightly approach. Essentially by construction, $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Notice that also

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \mathrm{R}^{\pm}} \mathfrak{g}_{\alpha}$$

defines two subalgebras of $\mathfrak{g}$. Finally, we let $\mathrm{b} = \mathfrak{h} \oplus \mathfrak{n}^{+}$ and call it a *Borel* subalgebra.

Let F_{μ} be a one-dimensional vector space generated by a vector v_0. We define a b-representation by declaring

- $\pi(\mathrm{H})v_0 = \mu(\mathrm{H})v_0$ for all $\mathrm{H} \in \mathfrak{h}$
- $\pi(\mathrm{X})v_0 = 0$ for all $\mathrm{X} \in \mathfrak{n}^{+}$.

In other words, F_{μ} is a $\mathrm{U}(\mathrm{b})$-module where H is acting by $\mu(\mathrm{H})$ and X acts by zero. Now, the inclusion $\mathrm{b} \to \mathfrak{g}$ induces a map of algebra $\mathrm{U}(\mathrm{b}) \to \mathrm{U}(\mathfrak{g})$. We then define the *Verma module* W_{μ} to be

$$\mathrm{W}_{\mu} = \mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathrm{b})} \mathrm{F}_{\mu}$$

which, by construction, is a left $\mathrm{U}(\mathfrak{g})$-module. In other words, we have constructed our representation of $\mathfrak{g}$.

Definition 3.1. A $\mathfrak{g}$-rep V is *highest weight cyclic with highest weight* $\mu \in \mathfrak{h}$ if there is $0 \neq v \in \mathrm{V}$ such that

- $(\pi\mathrm{H})v = \langle \mu, \mathrm{H} \rangle v$, for all $\mathrm{H} \in \mathfrak{h}$
- $(\pi\mathrm{X})v = 0$, for all $\mathrm{X} \in \mathfrak{g}_{\alpha}$, with $\alpha \in \mathrm{R}^{+}$
- the smallest invariant subspace containing v is V

Theorem 3.2. The Verma module is a highest weight cyclic representation with highest weight μ (and μ-weight vector $1 \otimes v_0$).

Moreover, W_{μ} satisfies the following universal property: any highest weight cyclic $\mathfrak{g}$-representation V with highest weight μ is a quotient $\mathrm{W}_{\mu} \twoheadrightarrow \mathrm{V}$ of the Verma module.

The hard thing to check here is the fact that W_{μ} is indeed cyclic. Here enters a big sledgehammer.

Theorem 3.3 (PBW). Let $\mathfrak{g}$ be a Lie algebra with ordered basis $x_1, \ldots, x_k$. Then the elements $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$, as the $n_i \geq 0$ vary, form a basis (called a PBW-basis) of $\mathrm{U}(\mathfrak{g})$.

Suppose $\alpha_1, \ldots, \alpha_{\mathrm{N}}$ are an ordering of the positive roots R^{+}, with corresponding root vectors $\mathrm{X}_{\alpha_i}, \mathrm{Y}_{\alpha_i}$, with X positive and Y negative. Let $\mathrm{H}_1, \ldots, \mathrm{H}_r$ the elements of our fixed base Δ. The collection $\mathrm{Y}_1, \ldots, \mathrm{Y}_{\mathrm{N}}, \mathrm{H}_1, \ldots, \mathrm{H}_r, \mathrm{X}_1, \ldots, \mathrm{X}_{\mathrm{N}}$ form a basis for $\mathfrak{g}$.

The corresponding PBW-basis give us a recipe to work out a basis for the Verma module W_{μ}, since we know exactly how the H_i and the X_i act on v_0. Indeed, one sees that W_{μ} is isomorphic to $\mathrm{U}(\mathfrak{n}^{-})$, as vector spaces.

4. Quotients

This is great so far. But how do we construct our coveted *finite-dimensional* irrep with highest weight μ? [which was promised to us by the theorem of the highest weight] Once again, notice that a $\mathfrak{g}$-representation is the same as a $\mathrm{U}(\mathfrak{g})$-module. Moreover, if V is a $\mathfrak{g}$-representation, then an invariant subspace is the same as a submodule! So, an irrep is a representation with no non-trivial invariant subspaces. So we need to find a $\mathrm{U}(\mathfrak{g})$-module with no non-trivial submodules.

It turns out that W_μ has a unique ***maximal*** submodule. Its quotient, which we call V_μ, is therefore an irreducible representation of $\mathfrak{g}$. Finally, one shows that when μ is dominant and integral, the quotient V_μ is actually finite-dimensional. Loosely speaking, the reason is the following. As a vector space, W_μ is the direct sum of its weight spaces. Each weight space is finite-dimensional, but there might be infinitely many of them. Passing to the quotient V_μ, the same is true. Moreover, the set of weights for V_μ is invariant under the action of the Weyl group. If μ is dominant integral, then any weight must also be integral. In particular, if μ' is a weight, then $w\mu' \leq \mu$ for all $w \in W$. By integrality, there are only finitely many possibilities for μ'.

5. $\mathfrak{sl}_2$

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbf{C})$. The universal enveloping algebra $U = U(\mathfrak{g})$ is the associative algebra on generators x, y, h subject to the relations given (as a two-sided ideal) by

$$hx - xh = 2x$$
$$hy - yh = -2y$$
$$xy - yx = h.$$

Let $\mu \in \mathbf{C}$. The Verma module W_μ is then the quotient of U by the left-ideal generated by

$$x = 0$$
$$h = \mu$$

where by μ we mean $\mu \cdot 1$.

Proposition 5.1. As a vector space, W_μ is isomorphic to $\mathbf{C}[y]$.

Indeed, $\mathfrak{sl}_2$ is generated by the three matrices X, Y, H. Its Cartan $\mathfrak{h}$ is the span of H, while $\mathfrak{n}^+$ is the span X, and $\mathfrak{n}^-$ is the span of Y. So $W_\mu = \mathbf{C}[y] = U(\mathfrak{n}^-)$ as vector spaces.

To conform to perhaps more standard notation, we write $v_0 = 1$, $v_1 = y$, $v_2 = y^2$, and so on. Let us now see what W_μ looks like as a $\mathfrak{g}$-module.

The easiest to compute is the action of y. We have

$$y \cdot v_j = v_{j+1}$$

which follows from the fact that $yy^j = y^{j+1}$. Next, let us have a look at the action of h.

$$h \cdot v_0 = h \cdot 1 = h = \mu = \mu v_0$$

so, as expected, v_0 is a weight vector with weight μ. Without being careful, one is lead to conclude the following:

$$\begin{aligned} h \cdot v_1 &= h \cdot y \\ &= \mu \cdot y \\ &= \mu y \\ &= \mu v_1 \end{aligned}$$

so that y_1 would also be a μ-weight vector. But this is ***WRONG!*** The problem lies in the second equality: substituting h for μ is ***not*** allowed. The reason is that h is playing the role of a scalar, and scalars live in $\mathrm{U}(\mathfrak{g})$, where the relation $h = \mu$ does not exist. The correct string of equalities is as follows:

$$\begin{aligned} h \cdot v_1 &= h \cdot y \\ &= hy \cdot 1 \\ &= hy \cdot v_0 \\ &= (yh - 2y) \cdot v_0 \\ &= y \cdot h - 2y \\ &= y \cdot \mu - 2y \\ &= (\mu - 2)y \\ &= (\mu - 2)v_1 \end{aligned}$$

so v_1 is a weight vector with weight $\mu - 2$. After the sixth equal sign, the substitution $h = \mu$ *is* allowed, as h is now seen as an element of W_μ, where that relation exists. Let's do one more:

$$\begin{aligned} h \cdot v_2 &= h \cdot y^2 \\ &= hy \cdot v_1 \\ &= (yh - 2y) \cdot v_1 \\ &= y \cdot (h \cdot v_1) - 2y \cdot v_1 \\ &= y \cdot (\mu - 2)v_1 - 2v_2 \\ &= (\mu - 2)y \cdot v_1 - 2v_2 \\ &= (\mu - 2 - 2)v_2 \end{aligned}$$

so that v_2 is a weight vector with weight $\mu - 4$. In general,

$$h \cdot v_j = (\mu - 2j)v_j.$$

By a similar analysis, we have

$$x \cdot v_j = j(\mu - (j-1))v_{j-1}$$

where we set $v_{-1} = 0$.

Now, if $\mu \notin \mathbf{Z}_{\geq 0}$ one sees that W_μ is irreducible. On the other hand, if $\mu \in \mathbf{Z}_{\geq 0}$ then the subspace

$$\mathrm{S} = \mathrm{Span}\{v_{\mu+1}, v_{\mu+2}, v_{\mu+3}, \cdots\}$$

is actually a maximal invariant subspace. Why is that? Well, the action of y just pushes vectors forward, the action of h is diagonal, but what about x?

$$x \cdot v_{\mu+1} = (\mu+1)(\mu - (\mu + 1 - 1))v_\mu = 0.$$

Thus, S is invariant (and also maximally so), and $\mathrm{V}_\mu = \mathrm{W}_\mu/\mathrm{S}$ is the desired (finite dimensional!) irrep of $\mathfrak{sl}_2$ with highest weight μ.